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CONTENTS

Mineralogy. — "*On the Mineral Component of Bones*", By W. F. DE JONG. (Communicated by Prof. G. A. F. MOLENGRAAFF.)

(Communicated at the meeting of March 27, 1926).

The chemical analysis of recent bones shows that the proportion in which they contain mineral constituents is approximately constant; f. i. the composition of a human thigh-bone is according to CARNOT: [1])

$$Ca_3(PO_4)_2 \quad 87{,}45\ \%$$
$$CaCO_3 \quad 10{,}18\ \%,$$

and a varying amount of CaF_2, $CaCl_2$ etc. making up only a small percentage of the whole. Fossilisation raises the amount of F gradually and in palaeozoic bones it is about as high as in normal fluor-apatite.

The ratio $\frac{CaCO_3}{Ca_3(PO_4)_2}$ being found about the same everywhere [2]) it may readily be suspected that bones are built up chiefly by one compound.

In fact CARNOT has already suspected the occurrence of a compound $3\,Ca_3(PO_4)_2 \cdot CaCO_3$, i. e. a mineral of the apatite series, of which the general formula is:

$$3\,Ca_3(PO_4)_2 \cdot CaX_2; \qquad [X_2 = F_2, Cl_2, (OH)_2, O, SO_4, CO_3 \text{ etc.};$$

Ca can be substituted by the metals Fe, Mg etc. [3])]

The compound $3\,Ca_3(PO_4)_2 \cdot CaCO_3$ is not known as a pure, well-crystallized material [4]), but minerals of this composition are known by the name of podolite and dahllite [5]); both are poorly studied and often reported to contain water [6]).

ROGERS [7]) enlarges the scope of the conception "*collophanite*" and combines the whole analysis into one compound:

$$3\,Ca_3(PO_4)_2 \cdot [CaCO_3]_n \cdot [H_2O]_x,$$

(n ranging from 0.55—1.43 and x from 0.28—3.03), and describes it as being amorphous with an index of refraction of 1.573—1.621, a weak anomalous double-refraction and an anomalous pleochroism.

To get to the root of this question exclusively optic investigation is not sufficient. This is why we have endeavoured to approach a solution

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1) A. CARNOT, Recherches sur la composition générale et la teneur en fluor des os modernes et des os fossiles. Ann. des Mines, 1893 III p. 164.

2) Cf. also FRANCHET, Sur la dissolation des os et des dents des sépultures préhistoriques. Revue Anthropologique 1925 p. 34.

3) A. CARNOT. Ann. des Mines. Série 9, Tome X, p. 45.
A. F. ROGERS. Zt. f. Krist. 52, p. 213 (1913).

4) C. HINTZE, Handbuch der Mineralogie I p. 515.

5) C. DOELTER, Handbuch der Mineralchemie, Bd. III. Abt. I p. 352 and 577.

6) GROTH—MIELEITNER, Mineralogische Tabellen, 1921.

7) A. F. ROGERS, Mineralogy and petrography of fossil bones. Bull. of the Geol. Soc. of Am. Vol. 35 p. 535, 1924.

by means of röntgen-rays. Photos were taken of a number of pulverized bones with Cu_{K_α}-rays after the exposure-method of DEBIJE-SCHERRER [1]). The bones were fossil bones derived from the Limburg-Chalk, from the phosphatic deposits in Tunis [2]), from Ursus spelaeus (diluvium in Moravia) and from human bones exhumed from the graves in Hungary and Drenthe; besides some recent bones of a seal, etc.

These photos cannot be distinguished from those of apatite, though the lines may sometimes appear to be more or less hazy and broadened. For further comparison photos have also been taken of the minerals staffelite, osteolite and pyromorphite [$3\,Pb_3(PO_4)_2 \cdot PbCl_2$], which belong to the apatite series. These pictures are approximately the same. Collophan [3]), however, presents a totally different picture, as can be seen in the photometer-diagrams of Fig. 1, made with the self-recording contrivance devised by MOLL [4]).

We are, therefore, justified in concluding, that a mineral from the apatite-series preponderates in the composition of the bones. The facts that no lines are found implying the presence of other components [5]) and

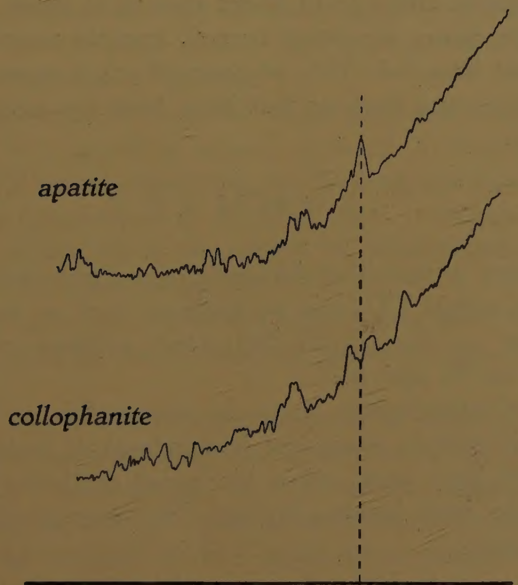


Fig. 1.
The lines on the films appear in the diagram as peaks. The place and the intensity of the lines of the two minerals deviate considerably.

1) DEBIJE and SCHERRER. Kgl. Ges. d. Wiss. Göttingen, Dez. 1915; Phys. Zt. 17, p. 277, 1916.

2) A. CARNOT. Ann. des Mines, Tome X, p. 200.

3) From the island Mona. Dana, The System of Mineralogy. 6 Ed. p. 108.

4) Proc. of the Phys. Soc. of London, 1921, p. 207.

5) Some percentage of an other crystalline substance is very difficult to detect by means of an analysis after the method we used. (Cf. WYCKOFF, The structure of crystals, 1924, p. 419), but it is not likely that a more or less considerable amount of $CaCO_3$ should not have yielded distinguishable lines.

that the ratio $\frac{CaCO_3}{Ca_3(PO_4)_2}$ in $3\,Ca_3\,(PO_4)_2 \,.\, CaCO_3$ coincides with the observation i.e. 9.7 : 90.3 (cf. p. 870), induce us to assume the presence chiefly of $3\,Ca_3\,(PO_4)_2 \,.\, CaCO_3$, or perhaps $3\,Ca_3\,(PO_4)_2 \,.\, CaO$.

It need not surprise us that no difference is noticeable between the röntgen-pictures of the hypothetical carbonate-apatite and the ordinary apatite. This may be due partly to the haziness and the faintness of various lines, but it will no doubt chiefly be caused by the great resemblance of the two molecules in which the radicals CO_3 etc. exercise but little influence upon the general structure.

The assumption of an apatite-mineral in bones is quite in keeping with the observations of ROGERS and others. The faint double-refraction, the minute pleochroism and the index of refraction, (F-apatite = 1.64—1,65), the solubility in HNO_3, the S.G. are entirely or approximately similar to those in the familiar terms of the apatite-series.

It is noteworthy that the crystals in fresh bones are so small that the röntgen-lines are hazy; so they do not contain more than, say, a hundred molecules. Most fossil bones also yield these lines. On the other hand while the bones are being burned, crystals accrue so that the lines appear sharply bounded. This perhaps affords a mean of differentiation between prehistorical findings that have been exposed to fire and those that have not.

SUMMARY.

Bones contain a mineral of the apatite-series, most probably $3Ca_3(PO_4)_2 \,.\, CaCO_3$. The radical CO_3 can be replaced here — as is often the case in fossilization — by F_2, Cl_2, SO_4 and perhaps others, Ca by Mg and perhaps by Fe and Al.

The mineral collophanite cannot be recognized.

The crystals contain some tens or hundreds of molecules, the röntgen-photo showing hazy lines; when the bones are being burned the crystals accrue and the lines become distinct. This may be of some importance for prehistorici.

The great significance of a research by means of röntgen-rays of bones was suggested to me by Dr. P. KRUIZINGA through whose interference I also obtained suitable material from Dr. A. E. VAN GIFFEN. Prof. J. A. GRUTTERINK allowed me free use of the Mineralogical Laboratory and gave me his valuable aid. To these gentlemen I desire to express my great obligations.

From the Mineralogical Laboratory of the Technical University of Delft.

Physiology. — *"Diet and reproduction"* II. By G. GRIJNS and K. DE HAAN.

(Communicated at the meeting of April 24, 1926.)

After our communication at the meeting in October a.p. we continued our investigations on the connection between nutrition and reproduction.

We still observed 4 male rats: 2 on maizfood II and 2 on maizfood III all second generation. All 4 were sterile. Thus we found till now among 14 males on the 3 maizfoods that only one was fertile. We also examined some gonads of these rats. They proved to differ much in development; some ones only weighing $^1/_6$ of normal testes other ones about $\frac{1}{2}$. In the smaller gonads the epithelial cells of the seminiferous tubules were almost totally vanished, and only a few cells rich in protoplasm remembered the spermatocytes. In the other ones spermatogonia, spermatocytes and spermatozoa were present and did not show many differences with those in normal testes. But in fresh smears they showed little mobility even after addition of a solution of glucose.

In an other experience we used an other ration viz. basal ration with yellow fat. This is a vegetable extract mixed to a vitamin free fat, on which we intend to report afterwards, as our investigations on it are not yet finished. The composition of this diet was: albumen 2, casein 15, hardened fat 12, yellow fat 3, rice starch 52, McCollums saltmixture 5, marmite 5, decitrated lemonsquash 5.

The young rats were put to this diet immediately after weaning; they grew well on it, even a bit faster than rats on normal diet, as may be seen from diagram 1.

The results of matings have been recorded in table 1. It shows that fertility, judged by the number of young rats born was very sufficient. Out of 11 matings 2 were sterile; in one case the course of the weight indicated resorption of young; once we only found blood in the cotton-wool of the case of a female we had separated as pregnant. In this case dead young had probably been eaten.

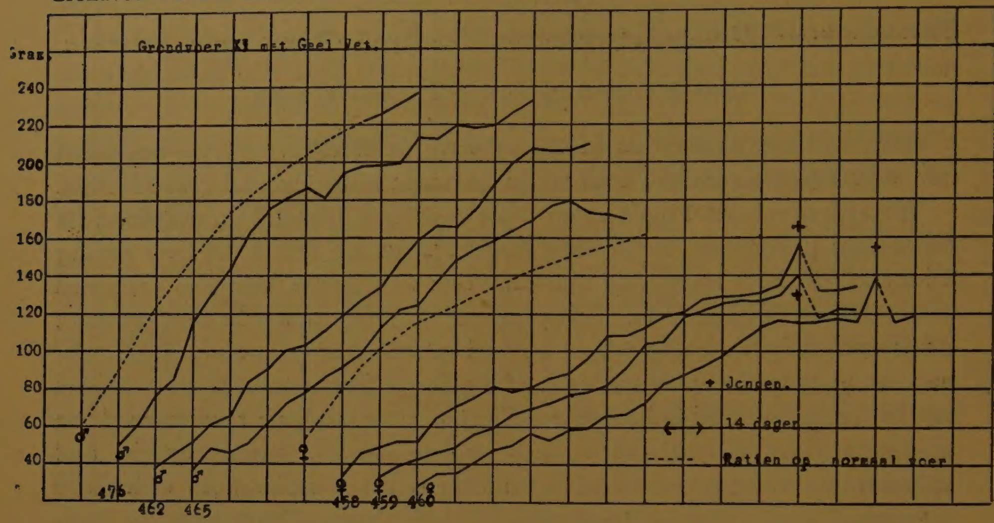
The remaining 6 matings yielded 38 youngs that died all except 2 a few days after birth. The mother of the 2 surviving (young of 460 and 483) had been given normal diet 3 days after whelping.

In a second experiment we used as main food whole wheat meal that had been exhausted with aceton for 10—12 days. According to BARNETT SURE's publications we supposed to be able to remove in this way the fertilising factor. Other vitamins however also being soluble in aceton, we had to supply these.

Therefore we composed a ration XIII as follows: Aceton extracted

whole wheat meal 250, wheat starch 125, hardened fat 60, butterfat 15, casein 25, decitrated lemonsquash 20, saltmixture as in cornfood 15. This

Grondvoer XI met Geel Vet = Basal diet XI with yellow fat.



Jongen = young. 14 dagen = 14 days. Ratten op normaal voer = Rats on normal diet.

Fig. 1.

TABLE 1.
Basal diet XI with yellow fat.

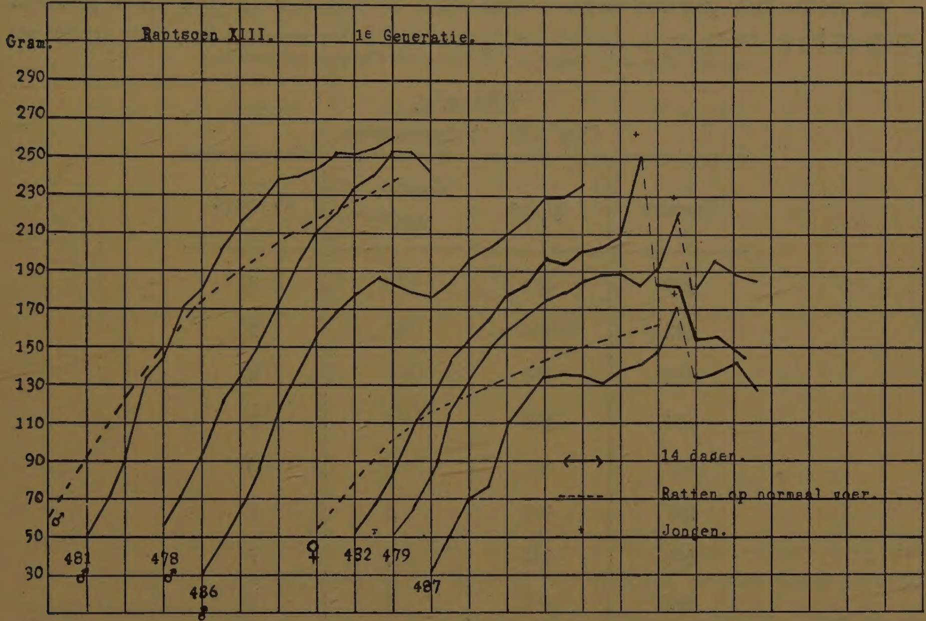
Females	Males	Number of young	Weaned
458	483	7	0
458	467	6	0
458	483	6	0
460	467	7	0
460	483	5	0
460	483	2	2 [1])
463	477	abortus	
463	477	dead young	
459	477	0	0
459	471	5	0
459	476	0	—

ration was also fed immediately after weaning. The rats grew very well, as may be taken from the curves in chart 2, where the growth of some at

[1]) Normal ration after whelping.

random rats is represented. From 8 matings (see table 2.) we got 51 young rats, 30 of which were weaned or 59 %.

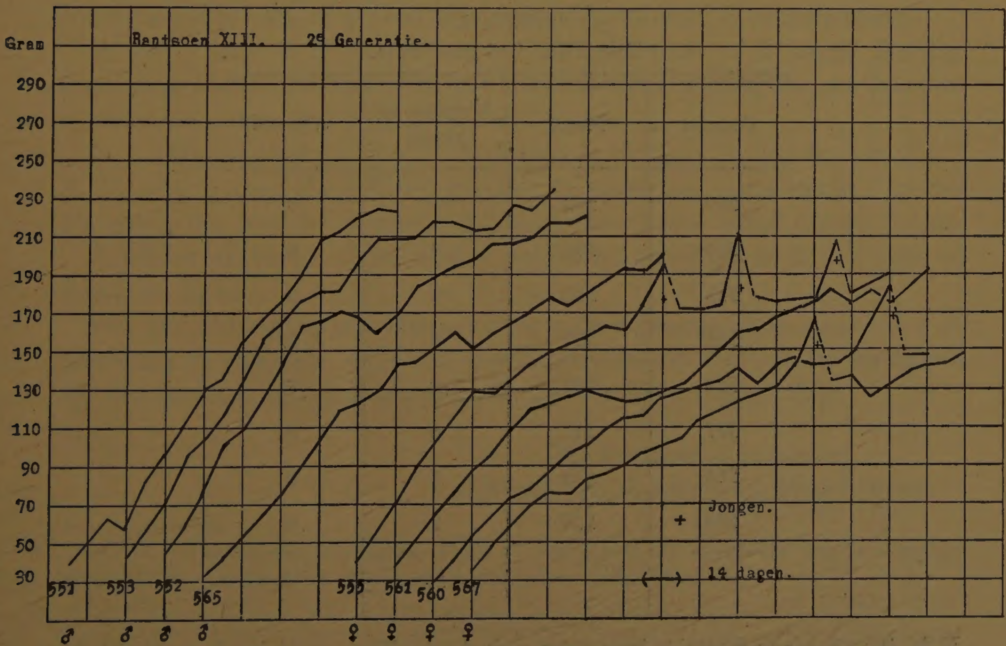
Rantsoen XIII = Ration XIII. 1e Generatie = 1st Generation.



14 dagen = 14 days. Ratten op normaal voer = Rats on normal diet. Jongen = young.

Fig. 2.

Rantsoen XIII = Ration XIII. 2e Generatie = 2nd Generation.



Jongen = young. 14 dagen = 14 days.

Fig. 3.

These were divided in 2 lots, the first getting the same diet as before, while for the second lot the whole wheat meal was not digested with aceton (ration XIIIa).

In both lots growth was normal. Of a number of the first growth curves have been reproduced in figure 3. Of the second lot, that grew as

TABLE 2.
Diet XIII. 1st generation.

Females	Males	Number of young	Weaned
479	489	9	3
482	486	10	9
484	478	7	0
484	486	4	3
487	480	2	0
487	486	3	0
488	481	8	7
497	458	8	8
		51	30

TABLE 3.
Diet XIII. 2nd generation.

Females	Males	Number of young	Weaned
550	565	6	0
555	565	9	0
555	565	6	0
555	553	6	0
560	552	5	0
567	552	8	3
568	553	8	0
568	553	7	0
		55	3

N.B. The 3 young from this litter grew very slowly. They could not be weaned before the seventh week. On normal diet this happens within 4 weeks.

fast, we added no curves to spare space. Table 3 shows the effect of matings in the first, table 4 of those in the second lot.

TABLE 4.
Diet XIIIa. 2nd generation.

Females	Males	Number of young	Weaned
549	558	1	0
554	563	9	0
554	563	5	0
564	557	abortus	
566	558	6	0
566	558	8	0
566	559	8	0

We see two rations, from which the first unables the mothers in the first generation immediately, the other in the second, to rear their young, while fertility in the males remains unchanged and that of the females, as far as to produce living young is considered, is but very little hampered.

That the animals fed on diet XIIIa were unable to rear their young, astonished us, for American investigations, we reported in our former communication, are in favour of a fertilising vitamin in wheatembryo.

In every case it is clear, that we can not speak of a single fertility controling factor, but that different foodstuffs are wanted for the function of the testis, than for that of the milkglands.

In a minute investigation on histological changes in sterile rats KARL E. MASON [1]) used the following ration to make male rats sterile: casein 18, starch 54, lard 15, butterfat 9, saltmixture 4 plus yeast tablets 0.4 gr.

Addition of 40 gr. fresh lettuce prevented sterility, as in the case of BARNETT SURE.

We are continuing our observations with different diets, to see if it is necessary to postulate still other specific substances that are indispensable for the function of other organs playing a role in reproduction.

Physiological Laboratory
Agricultural Academy of Wageningen.

[1]) K. E. MASON: A histological study of sterility in the albino rat due to a dietary deficiency. Proceedings Nat. Acad. of Sciences July 1925. Vol. 11.

Physiology. — *"On After-images and contrast in coloured light."* By Prof. G. GRIJNS and K. DE HAAN.

(Communicated at the meeting of May 29, 1926).

It is a remarkable fact, that half the century, that passed since HERING defended his „Gegenfarbentheorie" against the objections of DONDERS, did not bring a decision between this and YOUNG-HELMHOLTZ-MAXWELL's theory of complementary colours. Both have their partisans among physiologists and oculists, and all the work done on light- and coloursense has not been able to carry one of them to victory. Yet the adversary principles of those two views make one expect the possibility of finding such facts as are only compatible with one of them. HERING supposing an invertable reaction to be the cause of seeing colours (white and black included) one direction being called „Assimilation", the other „Dissimilation", the first associated with one sensation (black, green or blue) the other with the antagonistic sensation (white, red or yellow), seeks the explanation for the after-images in processes occurring in the visual substance (Sehsubstanz) involved by the preceding stimulus, while the assimilation-dissimilation rate that was removed from its rest-equilibrium returns to it, the opposite reaction prevails then and provokes the antagonistic sensation (Gegenfarbe). The colours of the complimentary after-image originate therefore in the „Sehsubstanz" itself, and must prevail, whereever such after-images appear.

The theory of YOUNG-HELMHOLTZ-MAXWELL on the contrary supposes three independent fundamental colours. It explains after-images by the intervention of fatigue after contemplating a colour for some time, wherefrom the other colours of the background prevail. According to this view the colour of the after-image is determined by that of the background.

HERING explains the contrast by induction. A beam striking a part of the "Sehsubstanz" and causing there assimilation, originates dissimilation in the adjacent parts and the reverse. How the action of irritated parts on not irritated ones occurs, HERING does not indicate. It can not be diffusion of dissociationproducts, as some of his partisans have suggested, because there would not remain matter for the after-images, and the immediate appearance of the contrast makes diffusion, which would require time, unacceptable.

HELMHOLTZ counts contrast to the erroneous judgments for which pleads, that it is most obvious, under circumstances, which suggest a coloured illumination, in which objects that are seen white ought to have the contrasting colour.

According to HERING's theory the contrast as well as the after-images always

must show the "Gegenfarbe", be it not unmixed under certain circumstances. While according to YOUNG-HELMHOLTZ's view colour of after-images and contrast will be destined by the difference of the colour present in the irritating light and the background.

Firstly we will consider those after-images, that originate if we fix for some time a coloured surface on grey paper, and without moving the eyes, remove the colour. But we use coloured light instead of white.

One of us [1]) published observations on after-images in monochromatic light, by which he showed them to be always black. We now extend them to polychromatic light and to contrast.

We used an electrical light of 100 candles, shut up in a light-tight ventilated box, with at one end an opening, in which we screwed a kind of nosepiece carrying the coloured glasses.

This illuminated a circle of about half a meter diameter on a grey paper, in which we placed the coloured paper, of which we wanted to study the after-images. All was placed in our dark room, so that only the filtered light entered the room.

Spectroscopically we found our lightfilters to let pass the following wavelengths:

1. Red	690—640	5 Bluish green	575—515
2. Orange	650—595	6 Blue	465—420
3. Yellow	690—510	7 Violet	500—420 and 690—660
4. Yellowgreen	570—540		

The papers used ordinarily have longer spectra and some white light mixed with the colour, but if we contemplate them in coloured light they only reflect such light as is common to paper and glass.

HELMHOLTZ [2]) experimented with coloured paper on coloured ground. He concludes that the after-images in those cases have a colour that is a mixture from that of the background and the complementary one of the paper.

We prefer to use coloured light, because colours that are not present in the ground are avoided and the white light from the background is excluded.

If we use white as image in all those experiments the after-image is black, as we might expect from our former investigations.

We thoroughly have discussed if we ought to use spectral light for composing our coloured light. This, however, would have necessitated a complex and expensive installation, and as we were free in choosing the

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1) G. GRIJNS: L'influence de la lumière colorée sur les images consécutives négatives. Arch. Néerl. de Physiol. VII p. 355, 1922.

2) H. VON HELMHOLTZ: Handbuch der Physiologischen Optik. Vol. II. p. 204. See also C. HESS, Unters. über die nach kurzdauernder Reizung des Sehorgans auftretenden Nachbilder. Pflügers Arch. 49 p. 201, 1891.

combinations of colours we were to experiment with, we took gelatinous lightfilters whichs we examinated spectroscopically.

The results of our observations are given in table 1 where those on contrast are joined too.

For studying contrast we laid a grey disc on a coloured background or a coloured paper on a grey one, covered this with a thin transparent white paper and illuminated with one of the coloured lights. Contrast is very distinct then, and it makes no difference if the colour or the grey is in the midst. With yellow paper we did not observe contrasts, and ordinarily no coloured after-images, because we could not get a yellow paper or yellow stuff, that did not contain a large part of red and green. So it showed in orange or light green light the colour of this light, the after-image being dark, still no contrast appearing.

From the table you may see, that according to the light used we may see from red a green, a blue or a yellow after-image or contrast, from blue a red, a yellow or a green, and so on. In all cases examined the colour of the after-image and the contrast is that what remains, if we substract from the colour of the light the colour we see the disc in. We never discovered a trace of a colour, that was not present in the light offered. The result of these experiments is that in not a single case is shown anything of light sensation of antigonistic colour originating in the eye, or more centrally.

We now will discuss another group of after-images, that give rise to more trouble, because they are more difficult to observe and different investigators describe them differently. It is the images we see after exposition of the eye to light, if this is darkened afterwards. As the time of illumination is of great influence on the effect, we firstly will consider what happens after a short exposition. We can experiment like BRÜCKE [1]), by lighting up an object for a moment e.g. by an electric spark or a momentshutter or like PURKINJE [2]), who moved a small object through the visual field while a given point was fixed. In the first case, we must attend to the phenomena, that arrive after the illumination, in the other we see the phases, through which the irritationprocesses pass away, behind each other. We must be aware, however, that what we observe with these two methods does not want to be identical, because in the latter the inducing light is still in the visual field, and a priori you cannot know, if this will exert an influence on sensation, or on the judgment about what happens in our visual space.

HAMAKER [3]) resumes the results of precedent observators in this way: After a short illumination the following phenomena occur.

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1) F. BRÜCKE: Versuche über subjective Farben. Poggend. Annalen d. Physik und Chemie LXXXIV. 418, 1851.

2) J. PURKINJE. Beobachtungen und Versuche zur Physiologie der Sinnen. Bd. II. p. 110. Quoted from HAMAKER.

3) H. G. HAMAKER. Ueber Nachbilder nach momentaner Helligkeit. Onderz. Physiol. Lab. Utrecht, Reeks V, Deel I, 1899.

TABLE I.

Colour of the disc.		Colour of the light.						
		Red	Orange	Yellow	Yellowgreen	Blue-green	Blue	Violet
Red	image after-image contrast	red grey black grey black	red yellow yellow	red green green	black bright green bright green	black bright green bright green	black bright blue bright blue	red blue blue
Orange	image after-image contrast	red grey black grey black	orange dark black black	orange green no blue green	grey uncertain uncertain	grey bright green green	black bright blue blue	red blue blue
Yellow	image after-image contrast	red grey black grey black	yelloworange grey black grey black	yellow none none	greenish yell. violet black uncertain	yellow blue none	black bright blue bright blue	red blue blue
Light green	image after-image contrast	dull red none none	orange none none	green dark grey dark	green none none	green grey grey	blue none none	black bright violet bright violet
Dark green	image after-image contrast	black bright red bright red	black bright orange bright orange	green orange orange	green none none	yellowgreen grey grey	black bright blue bright blue	black bright violet bright violet
Blue	image after-image contrast	black bright red bright red	black bright orange orange	dark green yellow uncertain	blue bright green bright green	blue bright green uncertain	blue none none	blue red red
Blue a.	image after-image contrast			black green uncertain	grey bright green bright green			
Blue b.	image after-image contrast			dark red bright green uncertain	black green green			

1. Primary short lasting image, that stays a moment after light is removed.
2. Very short dark interval.
3. After-image of PURKINJE (Secondary image of BOSSCHA) brighter than surrounding, of nearly complementary colour; lasts about $\frac{1}{4}$ sec.
4. Dark interval.

5. Faint light image, lasting a few seconds, colour indistinct.
6. A negative complementarily coloured after-image, only observable in special circumstances.

This series has been augmented with two phases by DITTLER and EISENMEIER [1]). They described a positive, isochrome, very short lasting image between 1 and 2 separated from 1 by a short dark space. It was observed behind a moving slit in a moderately lighted room, or in dark after short adaptation.

As we intend to publish our investigations in extenso elsewhere we will be concise here.

We investigated in both ways. With the momentshutter, we mounted on a light-thight lamp, we were not able to see the after-images of DITTLER and EISENMEIER. Our shortest exposition was 1/90 second. Then the image of PURKINJE appeared so soon, that the dark interval was not always observable, but often it looked as if suddenly a veil of different colour arose from the figure, thereafter all vanished. A moment later the 5th phase appeared.

We operated with nearly white light (incandescent lamp) and put coloured discs on the illuminated surface. The room was quite dark. Generally the after-image of PURKINJE is described as complementary, but if one thoroughly reads the protocols of several investigators, this proves not true.

HAMAKER writes [2]) "Seine Färbung war nicht immer eine komplementäre" and in table VIII page 30 it is purple or red for all colours. DITTLER and EISENMEIER [3]) call it more or less distinctly complementary. "Jedenfalls macht es nie einen mit dem Reizlichte gleichfarbigen Eindruck." HELMHOLTZ describes how the after-image of a shortly seen spectrum projected on a screen shows a reddish white spot, that corresponds as to its situation with the colours from orange to indigo-blue. Also after contemplating momentary different colours a crimson hue develops. Crimson HELMHOLTZ [4]) defines as purple mixed with white.

Most investigators assure that the after-image of PURKINJE is not seen in the fovea centralis and that it is absent after red.

The colour of this after-image to our opinion may be best compared with that of lightning, if we contemplated white objects. It is bright with a little bluish or purplish hue, that some times varies a little more to blue, other times more to reddish. It lasts very short and after a dark interval the under 5 mentioned phase follows. If the primary light was feeble, it is almost imperceptible. It seems like a thin mist of very minute lightdust

[1]) R. DITTLER und J. EISENMEIER. Ueber das erste positive Nachbild nach kurzdauernder Reizung der Sehorgans, mittelst beweglicher Lichtquelle. Pflügers Arch. f. die Ges. Physiol. Bd. 126 p. 610, 1909.

[2]) l. c. p. 8.

[3]) l. c. p. 643.

[4]) HELMHOLTZ. Physiol. Optik. Bd. II p. 213, 1911.

you scarcely can ascribe a colour to, but composed as it seems of more or less yellowish and greenish small points. HESS [1]) gives a good reproduction. The contours of the image are always very vague and its dimensions vary.

We put coloured discs one time on a black, the other time on a grey background, and with the aid of a momentshutter and a light of about 100 candles we illuminated them. We found the after-image of PURKINJE to be black for red, orange-red and violet. With orange, yellow, green and blue they were always bluish, sometimes a bit purple. On a gray ground red-orange and dark blue also have black after-images with a bluish bright area round them.

In our experiments with moving light, we had a black disc revolving before a uniformly lighted surface. In the disc was a small hole, before which slits of 1 to 5 mM. width and 3 to 4 cM. height could be adapted as well as coloured glasses. Screens prevented light to hit the foremost surface. After green, yellow or blue we saw the satellite moving very distinctly. Also behind a white slit. The colour always varied between purplish and bluish, behind a green slit tending a little more to purple, behind a yellow one to blue. Behind a white slit comes a bluish satellite. Our violet glass, that let pass red and blue had a bluish satellite. Then comes a dark phase, and afterwards an often long tail of lightcloud, that with a bluish green glass has a faint purple hue.

With red and orange glass the after-image of PURKINJE stays out. We paid particular attention to the after-images described by DITTLER and EISENMEIER. As we could give to our slit a velocity of 5 till 60 cM. pro second, we thought they would be easily seen. This was not the case. On rotating fastly different colours made different phenomena. Yellow, green and blue showed an enlargement of the slit, in which the former part of it ordinarily seemed more saturated than the latter. A red that contained rays of wavelength longer than 680 did the same, but less strongly . The other red (690—640) divided in two rectangles parted by a dark space, the first was saturated red, the second vermillion. An orange (640—590) slit in orange and yellow, another one (655—605) in red and yellow. The purple glass showed, at sufficient speed, a foremost purple and an immediately following greyish blue part.

On account of the measured rotationspeed, when the parting line just appeared between the dividing coloured slits, we find a difference with the primary image of about 0.04 seconds. The after-image of PURKINJE is certainly 6 till 8 times later.

We will continue our investigations on the latter phenomenon, which from the time it appears must be connected with that described by DITTLER and EISENMEIER.

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[1]) C. HESS. Ueber das Abklingen der Erregung im Sehorgan nach kurzdauernder Reizung. Pflügers Arch. 95. p. 1, 1903.

If we contemplate all that has been communicated on the after-image of PURKINJE it is evident, that it is as a rule not complimentary to the original image, but that in experiments with moving light the coloured image has a little influence on the after-image. This may be explained for as well by HERING's as by YOUNG's theory. As for the rest it has a not strictly defined hue, that remembers lightning, it does not appear with red, orange and violet, and evidently neither in the fovea. In HESS' [1]) experiment with white stripes on dark ground the after-image (which I see bluish also) may seem to pass over the fovea with a curve, but it may be as well, that we complete the hiatus in the movement with our imagination, so as we do in the case of moving pictures.

We therefore hold with HAMAKER, that it originates in the rods, and not in the cones, but we do not consider it as an after-image, but as the primary rod-image. We therefore suppose in the rods a longer latent periode than in the cones. If the stimulus is short enough the cones will be returned to rest before the rods answer and the primary image is seen twice: first by the cones in its own colour, then by the rods, that truly have no distinctionability for colours, but whose image however must appear to us in some hue, if we compare it with the sensation of the cones. It would be mere accident, if it was just alike to white.

The lack of coloursense explains for the incertainty of the colourquality, that must be ascribed to it and the variability on different days.

Our conception explains also for HERING's experience [2]), in which behind a black stripe, that is moved over a white ground, follows a black after-image, without supposing that a darkening of some spot in the retina gives a dark after-image on the enlightened part of this membrane. Here too the second image is the image of the rods, that discriminate between light and dark. Also what time concerns, as far as I can estimate it by the velocity, with which the black stripe must be moved, this black image corresponds to that of PURKINJE.

If the after-images of PURKINJE are properly the primary sensations of the rods, then the colours we attribute to them cannot be dependent of a reaction in the "Sehsubstanz", but ought to be the results of a judgment not of a direct perception.

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[1]) C. HESS, l. c

[2]) E. HERING. Eine Methode zur Beobachtung und Zeitbestimmung des ersten positiven Nachbildes kleiner bewegter Objecten. Pflügers Arch 126, p. 609, 1909.

Chemistry. — *"The hydration of dissolved saccharose and the expression of the concentration in measuring the activity of ions."* By I. M. KOLTHOFF. (Communicated by Prof. H. R. KRUYT.)

(Communicated at the meeting of February 27, 1926)

1. In an investigation on the influence of neutral salts on the concentration of hydrogen ions of dilute hydrochloric acid, of which a later communication will follow, the difficulty was how to express the concentration of the substances dissolved. This concentration is often measured in grams or mols of dissolved substance per liter of solution; others express it in grams or mols of dissolved substance per 1000 gms. of solution. In the measurements of osmotic pressure and related quantities the concentration is expressed in grammes of substance per 100 gms. of solvent, or in mols dissolved in a constant number of mols of solvent. When *n* mols of substance are dissolved in *N* mols of solvent the concentration is

$$x = \frac{n}{N}.$$

Finally the concentration can also be expressed in mols of dissolved substance in mols of solution, in other words, the concentration is

$$x = \frac{n}{n + N}.$$

In interpreting the influence of the concentration in measurements of osmotic pressure A. FINDLAY [1]) prefers the latter way of expression, and N. BJERRUM [1]) too uses it in calculating the activities of ions.

When experimenting with dilute solutions, it is immaterial which way of expressing the concentration is preferred. When, however, we experiment with liquids, in which the number of mols of dissolved substance is of importance with respect to the number of mols of solvent, the way how the concentration is expressed is of great importance.

2. In order to find out what way of expression is the most rational in my investigation, I have studied the influence of non-electrolytes on the concentration of hydrogen ions in 0.01 mol of hydrochloric-acid per liter of solution. The influence of saccharose was more particularly investigated.

As will be seen from what follows, saccharose practically does not influence the activity of hydrogen ions at this dilution of acid.

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[1]) A. FINDLAY, „The osmotic pressure" cited from N. BJERRUM, Z. anorgan. allgem. Chem. **109**, 275 (1920).

In the first place a series of experiments was made in which the concentration of the hydrogen ions is measured in 0.01 N hydrochloric acid, with and without the presence of saccharose. The measurements have been made with the hydrogen-electrode in the Physiological Laboratory of the University. The n calomel-electrode was used as standard half cell. The results obtained are given in the following table. 1.75 N as well as 3.5 N potassium chloride was used as a salt-bridge between the hydrogen- and calomel-electrode. By applying the extrapolation method of BJERRUM the diffusion potential could in this way be more or less eliminated. It is to be noted that this method could not be applied in solutions which contain more than 1 mol of saccharose per liter. The concentrations are expressed in mols per liter.

Concentration of hydrogen ions in 0.01 N HCl in a solution of saccharose.

Composition of the liquid	E. M. F. at 18° against N. C. E.	p_H	$[H^+]$
0.01 mol HCl p. liter	0.4041	2.040	0.91×10^{-2}
0.01 mol HCl + 0.5 mol of saccharose p. l.	0.3924	1.838	$1.45 \times$ —
0.01 „ „ + 1 „ „ „ „	0.387	1.744	$1.80 \times$ —
0.01 „ „ + 1.5 „ „ „ „	0.380	1.623	$2.4 \times$ —

These experiments show that the concentration of hydrogen ions of a solution of hydrochloric-acid apparently increases very much when saccharose is present. I have been able to confirm this result also with colourindicators. Of the liquids mentioned in the table above, $[H^+]$ was also colorimetrically determined with thymolblue and tropaeoline 00 respectively as indicator. The colour was compared with freshly prepared solutions of hydrochloric-acid, which were obtained by diluting 0.1 N HCl.

Colorimetrically determined $[H^+]$ of 0.01 N HCl in solution of saccharose.

Composition of the liquid	$[H^+]$ on thymolblue	$[H^+]$ on tropaeoline 00
0.01 mol HCl per liter	1×10^{-2}	1×10^{-2}
0.01 mol HCl + 0.5 mol of saccharose p. l.	$1.4 \times$ —	$1.32 \times$ —
0.01 „ „ + 1 „ „ „ „	$1.8 \times$ —	$1.75 \times$ —
0.01 „ „ + 1.5 „ „ „ „	$2.5 \times$ —	$2.1 \times$ —
0.01 „ „ + 2 „ „ „ „	$5-5.5 \times$ —	

As is well known, thymolblue is an indicator which behaves as an acid, tropaeoline 00, on the contrary, as a very weak base. Yet, in the saccharose solution their indication of the concentration of the hydrogen ions is almost

identical. Generally the values found with thymolblue are a little higher than those with the other indicator. This is explained by the fact that the dissociation constant of either indicator is very little diminished by the saccharose. The result is that thymolblue indicates a reaction which is a little too acidic, tropaeoline 00, on the contrary, a reaction slightly less acidic than is actually the case.

We may further observe that the agreement between the electrometric and colorimetric determinations is very satisfactory. It is seen that the concentration of hydrogen ions regularly increases with increasing concentration of saccharose. In a solution, containing 2.5 mols of saccharose and 0.01 N of hydrochloric-acid, I found, with thymolblue, a concentration of hydrogen ions of even $\pm 1.5 \times 10^{-1}$, consequently 15 times greater than might be expected according to the concentration of hydrochloric-acid (expressed per liter).

There is a great element of uncertainty in electrometric determinations. In liquids, containing hydrochloric-acid only, the diffusion potential, against potassium-chloride solutions, is rather large, so that the extrapolated value of the E.M.F. is at least 1 millivolt inaccurate. It is very difficult to find, when repeating the experiments, an agreement of 1 millivolt with the solutions mentioned.

Therefore another series of measurements has been made under the same circumstances; the difference is that all the liquids also contain 0.09 mol of potassium chloride per liter. A liquid containing 0.01 N HCl and 0.09 N potassium chloride is recommended by S. P. L. SÖRENSEN as standard in electrometric determinations of hydrogen ions. In agreement with SÖRENSEN we found that, in such a liquid at 18°, a hydrogen electrode has a potential against the N calomel-electrode of 0.4040 ± 0.0001 Volt.

I have made the measurements not only with the hydrogen electrode but also with the quinhydron electrode at 18°, using the quinhydron electrode in the standard hydrochloric-acid mixture as reference electrode. In this case the measurements could be reproduced to 0.0001 Volt.

When we refer the E.M.F. to the standard hydrochloric mixture the calculation of the change of p_H and $[H^+]$ is very simple, for the saccharose solutions contain the same quantity of hydrochloric acid and potassium chloride as the standard mixture. When π is the E.M.F. measured, the change of p_H is equal to

$$\Delta\, p_H = \frac{\pi}{0{,}0577}\,(18^\circ).$$

Moreover I have colorimetrically determined $[H^+]$ in the mixtures, by comparing them with solutions of hydrochloric-acid in 0.09 N potassium-chloride. The addition of the latter salt is absolutely necessary, as otherwise greatly divergent values are found. The presence of a minute quantity of salt has a rather great influence on the intensity and the tint of the indicators. I hope to return to this part of the subject in a future paper.

Measurements with the hydrogenelectrode in 0.01 N HCl, 0.09 N KCl and saccharose.

Composition of the liquid	π with resp. to 0.01 N HCl + 0.09 N KCl	Decrease of p_H by sacch.	Increase of the [H+] by sacch.
0.01 N HCl + 0.09 N KCl + 0.5 mol of sacch.	6.1 ± 0.4 m. Volt	0.104 ± 0.01	1.27 × ± 0.03
0.01 „ „ + 0.09 „ „ + 1 „ „ „	11.6 ± 0.4 „ „	0.201	1.59 × ± 0.03
0.01 „ „ + 0.09 „ „ + 1.5 „ „ „	18.4 ± 0.4 „ „	0.319	2.08 × ± 0.03
The same with the quinhydronelectrode.			
0.01 N HCl + 0.09 N KCl + 0.5 mol of sacch.	5.8 ± 0.1 m. Volt	0.101	1.26 times
0.01 „ „ + 0.09 „ „ + 1 „ „ „	12.5 ± 0.1 „ „	0.217	1.65 „
0.01 „ „ + 0.09 „ „ + 1.5 „ „ „	19.2 ± 0.1 „ „	0.334	2.16 „

Below are given the values which are found colorimetrically.

Compositon of the liquid	Increase of the concentration of the hydrogen ions with	
	Thymolblue	Tropaeoline 00
0.01 N HCl + 0.09 N KCl + 0.5 mol of sacch.	1.29 times	1.25 times (not sharp)
0.01 „ „ + 0.09 „ „ + 1 „ „ „	1.65 „	± 1.5 „ „ „
0.01 „ „ + 0.09 „ „ + 1.5 „ „ „	2.4 „	no accurate determination possible

The values found with thymolblue are probably a little too high. For the rest the agreement between the results is very satisfactory. Summarizing we find that by the presence of the saccharose the activity of the hydrogen ions apparently is increased ; on an average we find :

in a solution with 0.5 mol of saccharose per liter an increase of 1.27 ×
„ „ „ „ 1 „ „ „ „ „ „ „ „ 1.60 ×
„ „ „ „ 1.5 „ „ „ „ „ „ „ „ 2.1 ×

Some objections may be made to these results ; these we shall discuss first :

1⁰. It is possible that the constants of the hydrogen- and quinhydron-electrode are changed by the saccharose so that the method of calculation is inaccurate. This, however, is beside the truth, for I have also made measurements in buffer-solutions of different composition. In these the concentration of hydrogen ions is fixed, and is controlled by the relations of the undissociated acid and its anion. The total concentration of the two components is of subordinate value. Now I observed that the saccharose in the buffer-mixtures practically has no influence on the concentration of the

hydrogen ions; the very small effect that was found points to a minute decrease of the dissociation constant of the acid, which occurs in the buffermixture.

2⁰. It is possible that the sensitivity for hydrogen ions of the indicators used is changed by the saccharose. Indeed, the saccharose has a very slight influence on the dissociation constant of the indicators. The change which might be due to this influence, however, is much smaller than the effect we have observed. Moreover, the indicator tropaeoline 00 becomes a little less sensitive, so that in the saccharose mixtures it indicates a $[H^+]$ which is a little too low.

3⁰. Saccharose itself is a very weak acid. The dissociation constant, however, is very small, about 2×10^{-13}, and the concentration of hydrogen ions of 1 or 1.5 N saccharose is of the order of only 10^{-6}.

On account of its acid nature the saccharose cannot increase the concentration of hydrogen ions of 0.01 N hydrochloric-acid.

The increase of the activity of the hydrogen ions in the presence of saccharose really exists. In the first instance we can explain this influence in two ways:

1⁰. It is admitted that hydrogen ions in aqueous solution are hydrated and that the electrometric determinations give us only the concentration of anhydrous ions.

It would therefore be possible to assume that the dissolved saccharose has a dehydrating influence on the hydrogen-ions, so that more anhydrous ions appear. Then we should also have to admit that the colour indicators are sensitive only for anhydrous hydrogen-ions.

If this explanation were correct we should have to find that the saccharose would increase the concentration of hydrogen-ions in buffer-mixtures as much as in hydrochloric-acid. As is mentioned already this is not the case.

In solutions of organic acids saccharose has another effect than in hydrochloric-acid. We have found that 1 mol saccharose per liter increases the concentration of hydrogen-ions of hydrochloric acid $1.6 \times$. In solutions of organic-acids $[H^+]$ also increases, and this not $1.6 \times$, but less, namely so much as corresponds with an increase of the total concentration of 1.6 times.

2⁰. The other explanation which remains is that the saccharose displaces the water in the solution; the concentration of the water, or, the free volume of the water, diminishes, and the concentration of the solute increases. This would lead us to deduce that the concentration of the acid must be expressed in mols of solute, dissolved in mols of solution or mols of solvent. In our case there is little difference between the two modes of expression; experimentally we cannot decide which is the correct one. It is obvious to express the concentration in mols of solution.

I could also prove another way that one of the two modes of expression is correct.

As is said above, saccharose increases the concentration of hydrogen-ions in weak acids. Carbonic-acid, however, behaves differently; saccharose slightly diminishes the concentration of H-ions. The reason is, that in a carbonic-acid solution the greater part of the acid is present in the form of CO_2 (about 99 %), and that a small part is in the acid form as H_2CO_3. The concentration of the latter is also dependent of the concentration of the water:

$$CO_2 + H_2O \rightleftarrows H_2CO_3$$
$$[H_2CO_3] = K\,[CO_2]\,[H_2O].$$

If the concentration of the water becomes a $\times$ smaller, $[H_2CO_3]$ also becomes a $\times$ smaller. On the other hand the total concentration of the acid becomes as many times greater by the displacement of the water by the saccharose. If therefore the saccharose has no influence on the dissociation constant of carbonic-acid, we should find that the reaction is not changed by the saccharose. We indeed find a slight decrease of the concentration of the hydrogen-ions, corresponding with a slight fall of the constant.

The difference between saccharose and a substance as aethylalcohol is very peculiar as regards their influence on the dissociation constant of weak acids. The latter is strongly diminished by aethylalcohol, but hardly by saccharose. The explanation probably is that solutions of saccharose in water have a rather great dielectric constant [1]). In this respect saccharose, as a solvent, may be compared with water.

3⁰. If, from what precedes, we conclude that the concentration of the solute must be expressed as $\frac{n}{N}$ (or as $\frac{n}{n+N}$) in which n is the number of mols of solute and N the number of mols of solution, we can calculate from the increase of the concentration of the hydrogen-ions, how great the free volume is of the remaining water.

For, if we find that in 1 liter of solution, containing 0.01 mol of saccharose, the increase is 1.28 $\times$, this means we have 0.01 mol of HCl in $\frac{1000}{1.28} = 780$ C. of water. According to the specific gravity of a saccharose solution of 0.5 molal (WINDISCH's table) the liquid contains 895 cc of water per liter. From this we may conclude that 0.5 mol of saccharose binds 895—780 = 115 cC of water.

If we consider this water as hydration water, it follows, from what is said above, that in 0.5 molal solution of saccharose 230 gms. of water are bound by 1 mol saccharose, corresponding to a hydration of 12.8 mols of water for 1 mol of saccharose.

On account of the uncertainty in the measurements, mentioned in sub. 2, this last figure is not very certain, and changes between 10 and 14 mols.

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[1]) Compare i. a. THWING, Z. Physik. Chem. 14, 292 (1894).
DRUDE, Z. Physik. Chem. 23, 305 (1897).

From the results of the measurements with the other saccharose solutions we can also calculate the hydration, and then we find :

Hydration of saccharose in solution.

Concentration of saccharose per liter	Mols of water per mol. of saccharose
0.5 molal	10 to 11
1 „	6.6 „ 11.2
(1.5 „	9.2 „ 10.8)

As a slight error in the determinations of the concentration of the hydrogen-ions causes a great change in the hydration calculated, the latter cannot be deduced from the measurements with great certainty. We can only give the order of magnitude approximately (about 10 mols of water per mol of saccharose), which of course will diminish in concentrated solutions.

4. If saccharose apparently increases the activity of hydrogen-ions in solutions of hydrochloric acid, this must also manifest itself when the inversion constant in concentrated solutions of saccharose is determined.

Data in the literature prove indeed that the constant increases with increasing saccharose content. J. SPOHR [1]) already gave an explanation for this anomaly, by pointing out that the concentration should not be expressed in gms. of saccharose per liter of acid solution, but per gms. of solution. If this is done we get, according to SPOHR, a constant which is independent of the concentration of the saccharose dissolved.

ERNST COHEN [2]) formulated the deviation found more clearly. According to him, the volume of the saccharose in solution should be deducted, in order to calculate the exact concentration of the acid. For, considered kinetically, in a stronger saccharose solution the space, in which hydrochloric-acid and saccharose can collide, is smaller than when the volume of saccharose is small. Therefore the constant found in a stronger saccharose solution will be greater than in a more diluted solution.

M. A. ROSANOFF, R. H. CLARK, and R. L. SIBLEY [3]) demonstrated that the inversion constant really becomes independent of the concentration of the saccharose dissolved, when the concentration of the acid is expressed in the quantity of water present.

ERNST COHEN assumes that the volume of the saccharose dissolved is independent of the solution. In this way he gets the equation :

$$k_{40} : k_{20} = \frac{1}{100 - b_{40}} : \frac{1}{100 - b_{20}}.$$

Here k_{20} and k_{40} represent the reaction constants in a saccharose solution

1) J. SPOHR, J. prakt. Chem. **33**, 266 (1886).
2) ERNST COHEN, Z. Physik. Chem. **23**, 442 (1897).
3) ROSANOFF, CLARK and SIBLEY, J. Amer. Chem. Soc. **33**, 1911 (1911).

of 40 %, b_{40} is the volume of the dissolved saccharose in a solution of 40 %, b_{20} in a solution of 20 %.

In this way he calculates that 200 gms. of dissolved saccharose occupy a volume of 177 cc. According to the specific gravity this figure is 123 cc. Hence 200 gms. of saccharose bind 54 gms. of water, or 1 mol of saccharose 5.1 mols of water.

I have compared the values of the inversion constant found by W. OSTWALD [1]) and by J. SPOHR (l.c.) and moreover graphically deduced the value for a solution containing 0 % saccharose.

The figures are found in the following table:

Inversion constant at 25° according to OSTWALD, resp. SPOHR.

Concentration of sacch. p. 100 Cc	Constant of OSTWALD	of SPOHR
0 %	18.25	18.25
2 „	—	(20.47)
4 „	19.19	—
10 „	20.69	21.00
20 „	22.92	24.21
30 „	—	27.21
40	29.20	—

The value found by SPOHR in a solution of 2 % cannot possibly be good. For the rest the agreement between his values and OSTWALD's is not very fine. From the two series of observations I have calculated the inversion constant in a 0.5-, respectively 1 molal saccharose solution. At the same time is indicated how many times the inversion constant is increased, compared with the constant in a solution with 0 % of saccharose. In the two last columns the hydration of the saccharose has been calculated in the way as described above.

Hydration of saccharose according to inversion constant.

Concentration of saccharose	Inversion constant of		Increase of constant of		Hydration of sacchar. according to	
	OSTWALD	SPOHR	OSTWALD	SPOHR	OSTWALD	SPOHR
0.5 molal	22.25	23.4	1.22 ×	1.28 ×	8.3	12
1 „	26.90	28.4	1.47 ×	1.55 ×	6.2	8

In this way we therefore find for the hydration of the saccharose figures of the same order of magnitude as have been deduced from the increase of the concentration of the hydrogen-ions. Further we see from the last table

[1]) W. OSTWALD, J. prakt. Chem. 31, 316 (1885).

that the hydration decreases when the concentration of the saccharose increases.

5°. *Deduction of the hydration from other physical properties of solutions of saccharose.*

The relative decrease of the osmotic pressure by a dissolved substance is expressed by the equation :

$$\frac{p-p'}{p'}=\frac{n}{N}\left(\text{or }=\frac{n}{n+N}\right),$$

in which p and p' represent the osmotic pressure of the solution, respectively the solvent. From the measurements of the osmotic pressure of solutions of saccharose which have been made by H. N. MORSE and W. W. HOLLAND [1]) we cannot calculate the hydration with sufficient accuracy, as there is some irregularity in their figures. As this is the most accurate investigation on osmotic pressure of solutions of saccharose, we can draw from their figures this qualitative conclusion only, that the saccharose is hydrated.

A better result is reached with the data on the lowering of the vapour pressure, respectively the freezing-point of water by saccharose. Direct measurements of the vapour pressure of saccharose solutions have been made by A. SMITS [2]), who found the same figures as DIETERICI [3]).

Whereas the molecular lowering of the vapour pressure of saccharose in a very dilute solution is 0.084, it is 0.0902 in a solution of 1 mol of saccharose in 1000 gms. of water.

Hence the hydration of saccharose in the latter solution corresponds with 6 mols of water.

The most accurate measurements on the lowering of the freezing point of water by saccharose have been made by F. M. RAOULT [4]).

He found :

Grammes of sacch. in 100 g. of water	Molec. freezing-point depression
0	18.72
8.55	19.22
17.292	19.59
34.565	20.79

The molecular freezing-point depression increases with increasing concentration of saccharose ; we therefore get the impression that the

1) H. N. MORSE and W. W. HOLLAND, Amer. Chem. J. **41**, 1 (1909).
2) A. SMITS, Z. Physik. Chem. **39**, 385 (1902).
3) DIETERICI, Wied. Ann. **62**, 616 (1897).
4) F. M. RAOULT, Z. Physik. Chem. **27**, 617 (1898).

concentration of the saccharose increases more strongly than seems to correspond with the figures in the first column. If we assume that this is caused by the hydration of the saccharose, we find in the solution, which contains 34.565 gms. of saccharose, not 100 gms. of free water, but only

$$100 \times \frac{18{,}72}{20{,}79} = 90 \text{ gms.}$$

Hence we deduce a hydration of 5.5 mols of water, and by analogy 5.3 for a solution containing 0.5 mol of saccharose per 1000 gms. of water.

If we express the concentration as $\frac{n}{n+N}$ instead of $\frac{n}{N}$, we find a hydration of 6.4 for a solution of 1 mol of saccharose in 1000 gms. of water, and of 6.0 for a solution of 0.5 mol of saccharose in a similar amount of water.

Viscosity : According to A. EINSTEIN [1]) the relative viscosity of a very dilute solution is :

$$\mu = 1 + 2{,}5 \text{ v}.$$

Here v represents the volume of the solute. BURKHARD [2]) has determined the relative viscosity of saccharose at 20° with great accuracy, and has found a value of 1.0245 for a solution of 1 %. By means of EINSTEIN's equation and the specific gravity we calculate a hydration of 6.8 mols of water per 1 mol of saccharose. In this way we find a hydration of 8 for a solution of 2 % ($\mu = 1.0521$).

Lowering of the solubility of substances in water in the presence of saccharose.

According to NERNST [3]) the relative lowering of solubility on account of the presence of a foreign substance is equal to the number of molecules of that substance, divided by the number of molecules of solvent, therefore :

$$\frac{L - L'}{L'} = \frac{n}{N}.$$

So we have here a similar equation as for the lowering of the freezing-point and analogous magnitudes.

However we must apply NERNST's rule with the greatest circumspection, for it is based on the fact that the foreign substance — in our case saccharose — does not function as a solvent.

I have determined at 18° the solubility of different substances in water, and in saccharose solutions, and it appeared indeed, that the rule usually cannot be applied here.

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1) A. EINSTEIN, Ann. d. Physik. **24**, 34 (1911); Koll. Z. **27**, 137 (1912).

Originally EINSTEIN (Ann. de Physik. **19**, 301 (1906)) had deduced the equation: $\mu = 1 + \text{v}$, which, however, he has corrected later on.

2) BURKHARD, Z. des Vereins der Deutschen Zuckerindustrie **24**, 199.

3) W. NERNST, Z. Physik. Chem. **6**, 19 (1890).

Here are a few examples :

Concentration of saccharose per liter	Concentration of silver-acetate per liter	Relative decrease of solubility in %	
	Solubility of silveracetate.		
0	0.0745 molal		
0.5 molal	0.0745 „	0 %	
1 „	0.0740 „	0.7 „	
	Solubility of sulfanilic-acid.		
0 „	0.0614 molal		
0.5 „	0.0591 „	3.5 „	
1 „	0.0524 „	15 „	
	Solubility of salicylic-acid.		
0 „	0.0125 molal		
0.5 „	0.0131 „	— 4.5 „	
1 „	0.0134 „	— 7 „	
	Solubility of bitartrate of potassium.		
0 „	0.0269 molal		(S. ARRHENIUS)[1]
0.5 „	0.0243 „	10 „	(15 %
1 „	0.0208 „	22 „	20 „)

In the last table I have also given the figures for the lowering of solubility in potassium-acid-tartrate, which have been deduced from the observations of SV. ARRHENIUS. His results are not in agreement with mine.

From the figures found for the different substances we see that the relative lowering of the solubility on account of the presence of saccharose is not constant, but varies according to the nature of the substance. There is no doubt that the saccharose exercises a specific influence as a solvent.

It is to be expected that saccharose will have a much smaller specific influence on the solubility of gases, as the gases have a much smaller polar character than the substances mentioned in the table above.

In the following table I give a few results on the solubility of hydrogen (deduced from the determinations of P. STEINER [2]), and carbonic-acid (deduced from the determinations of A. CHRISTOFF [3])).

1) SV. ARRHENIUS, Z. Physik. Chem. **31**, 226 (1899).
2) P. STEINER, Wied. Ann. **52**, 275 (1894).
3) A. CHRISTOFF, Z. Physik. Chem. **53**, 329 (1905).

In the first column the concentration of saccharose is given in mols per liter, in the second column the calculated molecular depression of the solubility by saccharose, and in the third column the hydration derived from it.

Solubility of hydrogen in saccharose solutions (STEINER).

Concentration of sacch. in mols per liter	Molecular depression of the solubility in %	Calculated hydration of saccharose
1 molal	32 %	7 mols of water

Solubility of carbonic-acid in sacch. solutions (CHRISTOFF).

0.5 molal	42 %	12 mols of water
1 molal	33 %	7 mols of water

So we find here in 1 molal saccharose solution from the solubility of hydrogen, and of carbonic acid the same value for the hydration of saccharose.

6. *Discussion of the results.*

From the figures given for the hydration of saccharose we might suppose that the dissolved cane-sugar binds a certain number of molecules of water. This seems to me to be not the case. It is much better to assume that the saccharose possesses an affinity with respect to the molecules of water, which causes the latter to be "directed" in a certain position, by means of which the activity of the water, or the quantity of free water which remains, is lowered. It is not probable that all the molecules of the hydrate are in the same position; it is rather to be expected that the molecules of water, nearest to the saccharose, have lost most of their free mobility. The farther they are distant from the molecule of saccharose, the less they have lost of their activity, so that we get a gradual transition of the decrease of the activity of the directed molecules of water, nearest to the molecule of saccharose, to the free water in the solution. The effect which we observe, and which we call hydration, is therefore an expression for the total decrease of the free mobility of the molecules of water. In the preceding paragraphs we have calculated the hydration of the saccharose from different sorts of experiments. A summary has been given in the following table; here the hydration is given for a solution containing 1 mol of saccharose in 1 liter of solution.

Summary of the computed values of the hydration of saccharose (1 mol per liter)

Increase of [H+]	Increase of constant of inversion	Lowering of vapour pressure	Lowering of freezing-point	Viscosity	Lowering of solubility of gases
7—11	6—10	6	6.4	6.4	7

This table gives rise to the following observations:

a. The value calculated from the increase of the concentration of the hydrogen-ions of a dilute solution of hydrochloric-acid by saccharose is very uncertain. A small error in the measurement of the E.M.F. causes a great change in the value of the hydration. Moreover we have assumed in the calculation that the saccharose does not change the activity of the hydrogen-ions at all. If this activity should decrease by the influence of the saccharose, the values for the hydration would be too small. The average value of 9 mols of water per mol of saccharose is therefore to be considered as a minimum.

b. In the derivation from the increase of the inversion constant we have assumed that the viscosity of the medium has no influence on the reaction velocity. It may be expected, however, that the greater viscosity diminishes the chance of collision, and that therefore also the reaction velocity is somewhat decreased.

In this connexion I wish to point out that ERNST COHEN [1]) has made very accurate determinations on the inversion constant in solutions of hydrochloric-acid, in 20 % of alcohol. Here the alcohol has no influence as yet on the activity of the hydrogen-ions. Yet COHEN found a decrease of the constant of 4%, with respect to solutions free of alcohol, which perhaps must be attributed to an increase of the viscosity of the medium. However, too little is known of the influence of the viscosity on the reaction velocity to be very certain on this point. The mean value of the hydration of 8 is in any case to be considered as a minimum.

c. As was to be expected, the measurements of the lowering of the vapour pressure, respectively of the freezing-point of water, give the same value for the hydration of saccharose.

The calculation has been made on the assumption that the hydrate water has a vapour tension of 0. This seems improbable to me; for even in crystallized salts, with water of crystallization, the water has still a vapour pressure. This may also explain why, according to these two methods, we find too low a value for the hydration.

d. Neither is the value for the hydration, calculated from the viscosity, trustworthy. For EINSTEIN's equation, which is true for very dilute solutions only, may be applied only when the dissolved molecules are spheres, which have a large volume with respect to the molecules of the solvent among which they move.

Here too, therefore, we may consider as correct only the order of magnitude of the hydration calculated from the viscosity.

e. It has been observed already, that we have to be very careful with the explanation of the lowering of the solubility, in consequence of the presence of saccharose. Like glycerine, saccharose is also to be considered

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[1]) ERNST COHEN, Z. Physik. Chem. 28, 144 (1899), where other literature is also referred to.

as a solvent, and from the examples, given sub 5, we see that some substances dissolve better in the presence of saccharose than in pure water (salicylic acid; also benzoic acid, which is not mentioned in the table). Further it is also possible that the water of hydration has a dissolving action.

Determinations of solubility of substances that have little or no polarity, as carbon dioxyde, and hydrogen, are more reliable for this computation.

With the data given on this subject in the literature, we find a value for the hydration which agrees with the one found by other methods.

Summarizing we may say, that it is not possible to calculate the hydration accurately from physical data of saccharose solutions. Only the order of magnitude can be approximated, and then we come to the conclusion that a solution of 1 mol of saccharose per liter, contains 8 $\pm$ 2 mols of water as water of hydration.

Summary.

1. Sugar increases the concentration of hydrogen-ions of a solution of hydrochloric-acid.

2. This effect is explained by assuming that, when measuring the activity of ions, we must not express the concentration of the solute in mols per liter, but, as BJERRUM does, in mols of solute n per mols of solution $(n + N)$, therefore

$$x = \frac{n}{n + N}.$$

3. If we express the concentration in the way mentioned above, we find a slight increase of the activity of the hydrogen-ions by the presence of saccharose. This is to be attributed to the hydration of the dissolved saccharose.

4. The hydration of 1 mol of saccharose in 1 liter of solution has been calculated in different ways. We find an agreement in the order of magnitude, namely of 8 $\pm$ 2 mols of water per mol of saccharose.

5. The hydrate is not to be considered as a stoechiometric component of water and saccharose. It is probable that the molecules of water are directed by the saccharose in a certain position, so that the free mobility decreases. Hence the value which we calculate as hydration is in fact a measure for the decrease of the activity of the water by the dissolved saccharose.

Pharmaceutic Laboratory of the University.

Utrecht, December 1925.

Physics. — "*Prolegomena to a Theoretical Atomism*". By Prof. J. D. VAN DER WAALS JR.

(Communicated at the meeting of May 29, 1926).

Of late considerations have been developed by LOUIS DE BROGLIE [1]), EINSTEIN [2]) and SCHRÖDINGER [3]), in which it was tried to combine the existence of electrons, protons, and light quanta (which three quantities I will comprise under the name of "atoms") under one theoretic point of view. They assign to these atoms a pulsating phenomenon with the frequency $\nu = \frac{mc^2}{h}$. In this way the time-dependence of the phenomenon is determined, which dependence is more closely defined by DE BROGLIE, who assumes that for the electron that is not subjected to external forces, isophasia will prevail in space. But for the rest the extension in space is left undetermined. Now it seems to me that it is possible to get, in a natural way, a limitation in space for the isolated atom. For this purpose we describe the pulsating phenomenon as it presents itself to an observer who moves with it, by the aid of a function $e^{-\nu s}$, in which however we do not take for s the time, but the scalar quantity $\int \sqrt{dt^2 - \frac{1}{c^2}(dx^2 + dy^2 + dz^2)}$. This quantity is independent of the system of coordinates chosen, but dependent on the integration path chosen. In a ray of light which is emitted at a moment t_0 by an electron, it is this quantity that actually occurs under the sinus sign, in which as integration path is taken the "world-line" of the electron (hence also of the quantity of energy considered) up to the moment t_0, and further the world-line of the ray of light for which $\int ds = 0$.

This is valid for energy emitted by the electron. For energy belonging to the electron itself I would propose an integration path such that:

$$\int ds = t + ir.$$

It is easy to read from this formula what path is meant.

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1) LOUIS DE BROGLIE. Thèses etc. Paris. Masson et Cie 1924; also Ann. d. Physique X serie **3**, 22, 1925.

2) A. EINSTEIN: Berl. Sitzungsberichte 1925. pag. 3.

3) E. SCHRÖDINGER. Ann. d. Physik. **79**, 361 and 489, 1926. Phys. Zeitschr. 27 Jahrg. p. 95, 1926.

The pulsating phenomenon is then described by the function

$$e^{-\nu r}\,(\cos \nu t + \iota \sin \nu t)$$

and the same quantum hypothesis that determines the periodicity in time, determines at the same time the limited extension in space; and the reason that the atoms have a limited extension in spatial direction, but (at least on an average) an unlimited extension in time direction then lies in this, that the space-time continuum is not four dimensional, but one + three dimensional.

The above solution is not entirely satisfactory. The isophasia for a resting and non-accelerated atom is maintained. And it seems to me that serious objections may be advanced to this. DE BROGLIE discusses a mechanic analogon, in which a number of weigths hanging on springs executes isophasic vibrations. But in this it is clear that each of the weights performs its vibrations independent of the existence of the others. It behaves as a "windowless monade" of LEIBNITZ, and the isophasia is only possible through a "harmonia praestabilita". If a dynamic relation is supposed to exist between the different points participating in the pulsating phenomenon, progressive or stationary waves will sooner be assumed between them.

And for this reason I should prefer to determine the pulsating phenomenon by another function or choose another integration path for *ds*, if it appears to be possible to preserve at the same time the advantage of the natural restriction of the extension in space of the phenomenon through the quantum condition.

Mathematics. — "*A Quadruple Involution in Space*". By G. SCHAAKE. (Communicated by Prof. JAN DE VRIES).

(Communicated at the meeting of February 27, 1926).

§ 1. On p. 295 of Vol. XXI of these Proceedings Prof. JAN DE VRIES observed that the triple involution of REIJE in the plane may be produced by the aid of a net of cubics which has the angular points of a complete quadrilateral as base points. The triplets of the involution which may be derived from this net, are the triplets of the base points different from the said angular points of the pencils which are contained in the net.

In this communication we shall investigate a quadruple involution in space which may be produced in a similar way and which, also as regards its properties, forms an analogy to the triple involution of REIJE.

§ 2. We choose five planes $\alpha_1, \alpha_2, \ldots, \alpha_5$, and we indicate the first member of the equation of the plane α_i, with second member zero by a_i. Then

$$\sum \frac{\lambda_i}{a_i} = 0$$

where $\lambda_1, \ldots, \lambda_5$ are arbitrary constants, is the general equation of the biquadratic surfaces ω^4 which pass through the ten lines of intersection $\alpha_i \alpha_k$ of the planes α and which have a conical point in each of the ten points of intersection $\alpha_i \alpha_k \alpha_l$ of three of these planes.

Besides the ten lines $\alpha_i \alpha_k$ two surfaces ω^4 have a twisted curve k^6 of the sixth order in common, which passes through the ten points $\alpha_i \alpha_k \alpha_l$, as the complete intersection has a quadruple point in a point $\alpha_i \alpha_k \alpha_l$ and three lines $\alpha_i \alpha_k$ pass through such a point.

Besides along this straight line a plane through $\alpha_i \alpha_k$ cuts the two surfaces ω^4 along two cubics. These curves cut each other in the first place in the three points of intersection of the plane with the three lines of intersection of the planes α different from α_i and α_k. The remaining six points of intersection belong to k^6. Three of these six points lie on $\alpha_i \alpha_k$ in the three conical points of the surfaces ω^4 on this line, and three lie outside $\alpha_i \alpha_k$. Hence the curve k^6 does not cut a line $\alpha_i \alpha_k$ outside the three conical points of the surfaces ω^4 which lie on such a straight line.

A third surface ω^4 cuts k^6 in four points outside the ten points $\alpha_i \alpha_k \alpha_l$, in each of which two points of intersection of ω^4 and k^6 coincide. These four points are the base points of a net of surfaces ω^4.

Let C be a linear complex of surfaces ω^4. C contains ∞^3 nets of surfaces ω^4. The quadruple involution in question I^4 consists of the ∞^3 quadruplets of the base points of the said nets.

I^4 contains one quadruplet of which one of the points lies in a given point of space P. This is formed by the base points of the net of the surfaces of C which pass through P.

§ 3. Let us consider a four-dimensional space R_4 which contains the considered space R_3. Through each of the planes a_i we pass a three-dimensional space A_i lying in R_4. Any four of the spaces A_i cut each other in the points $H_1 \equiv A_2 A_3 A_4 A_5, \ldots, H_5 \equiv A_1 A_2 A_3 A_4$. If we choose the angular points of the pentahedron of coordinates in $H_1, \ldots, H_5$, the equation:

$$\sum \frac{\lambda_i}{x_i} = 0$$

is the equation of the biquadratic varieties Ω^4 which have triple points in $H_1, \ldots, H_5$. These varieties contain the ten planes $A_i A_k \equiv H_l H_m H_n$ and have double lines in the ten lines $A_i A_k A_l \equiv H_m H_n$. They cut R_3 along the surfaces ω^4 of § 2.

Besides the ten planes $A_i A_k$ two varieties Ω^4 have a surface ω^6 of the sixth degree in common. This surface has a triple point in any point H_i, as the complete intersection must have a nine-fold point in a point H_i and six planes belonging to this intersection pass through a point H_i. ω^6 passes through each edge $H_i H_k$, as this is quadruple for the complete intersection and contains three planes belonging to this intersection. The surface ω^6 cuts R_3 along a curve k^6 of § 2.

A third variety Ω^4 has in common with ω^6 besides the ten edges $H_i H_k$, each counted twice, a rational biquadratic curve k^4, which passes through the points H_i, because of the nine branches of the complete intersection which pass through a point H_i, each of the four straight lines $H_i H_k$ through this point splits off twice.

The linear complex C of surfaces ω^4 that has been assumed in § 2, may be considered as the intersection of R_3 and a linear complex Γ of varieties Ω^4 which, besides the points $H_1, \ldots, H_5$ have another point, H_6, in common. A quadruplet of I^4 consisting of the four common points of three surfaces ω^4 not belonging to a pencil, which are different from the ten points $a_i a_k a_l$, is the intersection of R_3 with a base curve k^4 of a net in Γ, which, therefore, passes through H_6. Now any curve k^4 is a base curve of a net of varieties Ω^4, as k^4 must have seventeen points of intersection with a variety Ω^4 through two points of k^4 different from $H_1, \ldots, H_5$, and, therefore, lies on this variety.

Accordingly the quadruple involution I^4 consists of the quadruplets of points of intersection of R_3 with the biquadratic curves which pass through the six points H_i in R_4.

§ 4. As R_4 contains ∞^{14} quadratic varieties, those which pass through the six points H_i form a linear system of ∞^8 individuals. The condition for a quadratic variety to contain a plane is six-fold because it is equivalent to the condition that the variety contains six points of this plane which do not lie on a conic. Consequently there are ∞^2 quadratic varieties which pass through the six points H_i and contain a given plane φ of R_3.

The intersection of two of these varieties consists of φ and a cubic surface which passes through the six points H_i and has a conic in common with φ as, besides φ, a space through φ has a plane in common with each of the two varieties, so that this space contains a straight line outside φ of the intersection of the varieties.

Besides the conic in φ the cubic surface has in common with a third variety containing φ which passes through the six points H_i, a biquadratic curve k^4 through the points H_i that cuts φ three times, because a space through φ which has also a straight line in common with the surface and a plane in common with the variety, contains one point of k^4 outside φ. Through this curve there pass all the quadratic varieties of the net which contain the six points H_i and φ.

The planes through a given point which cut the rational curve k^4 three times, form a quadratic hypercone K. For this hypercone intersects an arbitrary three-dimensional space in the surface of the trisecants of the projection of k^4 on this space out of the given point. Consequently the system of the ∞^2 quadratic varieties through the points H_i which contain φ, consists of hypercones K with vertices in φ. There is, therefore, only one curve k^4 which cuts φ three times.

Accordingly the quadruple involution I^4 contains one quadruplet of which three points lie in a given plane.

The surfaces ω^4 of the complex C cut a plane φ along a complex of biquadratic curves which pass through the angular points of a complete pentalateral. The triplet of I^4 in φ forms with the said angular points the system of base points of a net of biquadratic curves. Hence:

A linear complex of biquadratic curves all of which pass through the angular points of a complete pentalateral, contains one net with thirteen base points [1]).

§ 5. The system of the ∞^8 quadratic varieties which pass through the six points H_i, cuts R_3 in a linear system of ∞^8 quadratic surfaces. The ten coefficients a_{ik} of the equations of these surfaces always satisfy the relation:

$$\Sigma \lambda_{ik} \, a_{ik} = 0.$$

[1]) Prof. JAN DE VRIES drew my attention to the fact that these biquadratic curves are involution curves of involutions I^5 of the tangents to the conic which may be inscribed in the pentalateral and of which one group consists of the pentalateral.

Let us choose one variety out of this system which is degenerate in the planes $(\xi_1', \xi_2', \xi_3', \xi_4')$ and $(\xi_1'', \xi_2'', \xi_3'', \xi_4'')$. The coordinates of these planes satisfy the equation:

$$\Sigma\, \lambda_{ik} \cdot \tfrac{1}{2}\,(\xi_i'\, \xi_k'' + \xi_k'\, \xi_i'') = 0.$$

Hence these planes are associated to each other relative to a quadratic surface ω^2 which is represented by the equation in plane-coordinates:

$$\Sigma\, \lambda_{ik}\, \xi_i\, \xi_k = 0.$$

Accordingly those quadratic varieties of the net which contain the six points H_i and the plane φ of R_3, cut R_3 also in the planes of a sheaf that has the pole F of φ relative to ω^2 as vertex. An arbitrary plane φ' through F belongs entirely to that variety of the said net that contains two points of φ' which do not lie on a straight line with F. For besides the line $\varphi\varphi'$ this variety has three points in common with φ which do not lie on a straight line.

The point F is the intersection outside φ of the curve k^4 which is common to all the individuals of the net, and R_3. Consequently a point of intersection of a curve k^4 through the six points H_i and R_3 is the pole relative to ω^2 of the plane through the other three points of intersection of the same curve and R_3.

The quadruplets of the quadruple involution I^4 are, therefore, the systems of angular points of ∞^3 polar tetrahedrons of a quadratic surface ω^2.

§ 6. The joins $H_i\, H_k$ cut R_3 in the fifteen points D_{ik}, the planes $H_i H_k\, H_l$ along the twenty lines d_{ikl}, and the spaces $H_i H_k\, H_l H_m$ in the fifteen planes δ_{iklm}. The said points, lines and planes of R_3 form a configuration $\triangle\,(15_6, 20_3)$ of DESARGUES. The tetrahedrons $D_{13}\, D_{14}\, D_{15}\, D_{16}$ and $D_{23}\, D_{24}\, D_{25}\, D_{26}$ for instance are resp. the projections out of H_1 and H_2 of the tetrahedron $H_3\, H_4\, H_5\, H_6$ on R_3, so that the lines $D_{13}\, D_{23}$, $D_{14}\, D_{24}$ etc. all cut $H_1 H_2$ and, therefore, pass through D_{12}. Further the points of intersection $D_{34}, D_{35}, D_{36}, D_{45}, D_{46}, D_{56}$ of the corresponding edges lie in the plane δ_{3456} as well as the lines of intersection $d_{456}, d_{356}, d_{346}, d_{345}$ of the corresponding sides of the two above mentioned tetrahedrons of R_3. The planes $\alpha_1, \alpha_2, \ldots, \alpha_5$ of § 2 are resp. identical with the planes $\delta_{2345}, \delta_{1345}, \delta_{1245}, \delta_{1235}$ and δ_{1234}, i.e. the five planes without index 6. As we may exclude another index as well we find:

The quadruple involution I^4 may be produced in six ways as indicated in § 2.

If we choose one of the points P_1 of a quadruplet of I^4 in D_{12}, k^4 degenerates into the line $H_1 H_2$ and one of the ∞^2 twisted cubics which lie in the three-dimensional space $H_3 H_4 H_5 H_6$ and which pass through the points H_3, H_4, H_5, H_6 and the point of intersection S_{12} of the said

space with H_1H_2. These twisted cubics form a congruence of REYE [1]) and cut the plane δ_{3456} in a triple involution of REYE of which any triplet, together with D_{12}, forms a quadruplet of I^4.

The singular points of this triple involution, the points of a configuration (10_3) of DESARGUES, are the six points of $\triangle$ in δ_{3456} and the four points of intersection of δ_{3456} with the straight lines of $\triangle$ through D_{12}.

Accordingly the quadruple involution I^4 has cardinal points in the points D_{ik} of the configuration $\triangle$. To the point D_{ik} there correspond the point triplets of a triple involution of REYE *in the plane δ_{lmno}.*

As any quadruplet of I^4 forms the system of angular points of a polar tetrahedron of ω^2, the point D_{ik} must be the pole of the associated plane δ_{lmno} relative to ω^2. Hence the perspective tetrahedrons $D_{13}\,D_{14}\,D_{15}\,D_{16}$ and $D_{23}\,D_{24}\,D_{25}\,D_{26}$ are in a polar correspondence to each other relative to ω^2 and the center of perspectivity is the pole of the axial plane relative to ω^2. This holds good for all fifteen pairs of perspective tetrahedrons by the aid of which $\triangle$ may be produced.

If P_1 lies on the line d_{123}, k^4 degenerates into the conic through P_1 of the pencil of conics in the plane $H_1\,H_2\,H_3$ which has the points H_1, H_2, H_3, and the point of intersection S_{123} of the planes $H_1H_2H_3$ and $H_4\,H_5\,H_6$ as base points and into an arbitrary conic of the pencil of conics in the plane $H_4\,H_5\,H_6$ of which the points H_4, H_5, H_6 and S_{123} are the base points. The said pencils cut d_{123} and d_{456} resp. in the pairs of points of the involutions i_{123} and i_{456}. Accordingly P_1 belongs to ∞^1 quadruplets of I^4. Besides P_1 these quadruplets have one more fixed point, P_2, which is associated to P_1 in i_{123}. The other two points are variable and always form a pair of i_{456}.

Consequently the twenty straight lines d_{ikl} of $\triangle$ consist of points that are singular for I^4. The quadruplets of I^4 which contain a given point P_1 of d_{ikl}, consist of the point associated to P_1 in i_{ikl} and of a pair of the involution i_{mno} on d_{mno}.

As any quadruplet of I^4 consists of the angular points of a polar tetrahedron of ω^2, the two straight lines of each of the ten pairs (d_{ikl}, d_{mno}) are associated to each other relative to ω^2 and the twenty involutions i_{ikl} consist of the pairs of points of the corresponding straight lines d_{ikl} that are associated relative to ω^2.

§ 7. A curve k^4 is projected out of H_1 by a cubic cone $\varkappa_1{}^3$ which cuts R_3 along a cubic k^3 through the five points $D_{12}, D_{13}, D_{14}, D_{15}$, and D_{16}. Any cubic cone $\varkappa_1{}^3$ contains ∞^1 curves k^4 as the k^4 which passes through an arbitrary point of $\varkappa_1{}^3$, is projected on R_3 in a cubic which must have six points in common with the intersection $k_1{}^3$ of $\varkappa_1{}^3$ and R_3 and which, therefore, coincides with $k_1{}^3$.

The quadruplet of I^4 of which one point lies in a given point of a

[1]) Cf. the thesis for the doctorate of Dr. J. DE VRIES: *Bilineaire congruenties van kubische ruimtekrommen*, Utrecht 1917, cap. I.

curve k_1^3, belongs entirely to this curve as the k^4 through the given point lies on the cone which projects k^4 out of H_1, and which, therefore only cuts R_3 on k_1^3.

The curves in question k_1^3 form a congruence of REYE C_1, of which the base points are the angular points of a polar pentagon of ω^2, i.e. a skew pentagon so that each plane through three of the angular points is associated relative to ω^2 to the join of the other two angular points. The straight line for instance which is associated relative to ω^2 to $D_{12} D_{13} \equiv d_{123}$, i.e. the line d_{456}, lies in the plane $D_{14} D_{15} D_{16}$. And the pole of the plane $D_{14} D_{15} D_{16} \equiv \delta_{1456}$, i.e. D_{23}, also lies on d_{123}.

As we might as well have projected the curves k_1^4 out of $H_2, \ldots, H_6$ there are six such congruences of REYE, $C_1, \ldots, C_6$. Each of the congruences consists of the cubics containing ∞^1 inscribed polar tetrahedrons of ω^2 and I^4 is formed by the quadruplets of angular points of the ∞^3 inscribed polar tetrahedrons of the curves of one of these congruences.

Consequently the points P_2, P_3 and P_4 corresponding in I^4 to a given point P_1, may be found by determining the intersection of the polar plane of P_1 relative to ω^2 and the curve through P_1 of a congruence of REYE *of which the base points form a polar pentagon of ω^2. Thus I^4 may be produced in six ways.*

§ 8. The planes through the straight line $H_2 H_3$ which project a curve k^4, form a quadratic bicone K_{23}^2 which cuts R_3 in a cone $\varkappa_{23}^2$ with vertex in D_{23}. This cone contains the four lines of $\triangle$ through D_{23}, i.e. d_{123}, d_{234}, d_{235} and d_{236}. Any bicone K_{23}^2 contains ∞^2 curves k^4 as a k^4 through an arbitrary point of K_{23}^2 has two points of intersection with K_{23}^2 each counted double, i.e. the points H_2 and H_3, and five more single points of intersection, to wit the arbitrary point and the other four points H. A quadruplet of I^4 of which one point lies on the cone $\varkappa_{23}^2$, must entirely belong to $\varkappa_{23}^2$ because the k^4 through this point lies on K_{23}^2.

As this quadruplet also quite belongs to the curve k_1^3 through the same point, it consists of the four points of intersection outside D_{12} and D_{13} of this curve k_1^3 and $\varkappa_{23}^2$. For this reason the intersection of a cone $\varkappa_1^3$ and a bicone K_{23}^2 consists of the lines $H_1 H_2$ and $H_1 H_3$ and of a curve k^4.

Accordingly to each of the congruences of REYE *mentioned in § 7 there correspond ten pencils of quadratic cones which have their vertices in a point of $\triangle$ and of which the four straight lines through this point are generatrices so that I^4 consists of the quadruplets of the variable points of intersection of the curves of the congruence with the cones of each of the corresponding pencils. We have, therefore, found six congruences of* REYE *and fifteen pencils of cones by the aid of which I^4 may be produced in sixty ways.*

§ 9. The twisted cubics of the six congruences of REYE found in § 7, are invariant for I^4. The same holds good for the ∞^4 curves k^6 which are the base curves of the pencils of the complex C of surfaces ω^4 indicated in § 2. The surface of such a pencil which passes through a point of α_1, is degenerate in α_1 and a cubic surface that contains the lines of intersection of the planes $\alpha_2, \ldots, \alpha_5$ and has conical points in the points of intersection of any three of these planes. This surface has in common with the cubic surface which is a part of the surface of the pencil through an arbitrary point of α_2, the three straight lines $\alpha_3 \alpha_4$, $\alpha_4\alpha_5$ and $\alpha_3\alpha_5$ passing through the point $\alpha_3\alpha_4\alpha_5$ and a curve k^6 of which the said three lines are the only trisecants through $\alpha_3\alpha_4\alpha_5$. For both cubic surfaces have a conical point in $\alpha_3\alpha_4\alpha_5$; accordingly any trisecant of k^6 through $\alpha_3\alpha_4\alpha_5$ has four points in common with these surfaces and, therefore, belongs to them. Hence the projection of k^6 out of $\alpha_3\alpha_4\alpha_5$ on an arbitrary plane is a quintic with three double points; k^6 is, therefore, of the genus three.

As I^4 may be produced in six ways, as has been indicated in § 2, there are, accordingly, six systems of ∞^4 twisted curves of the order six and the genus three which are invariant for I^4.

§ 10. The curves k^4 which cut a given line l of R_3, form a surface. In the first place this surface has in common with the space $H_1H_2H_3H_4$ the cubic through the points $H_1, \ldots, H_4$ and the points of intersection of l and H_5H_6 with the said space, as together with H_5H_6 this curve forms a k^4 cutting l. The edges of the tetrahedron $H_1H_2H_3H_4$ also belong to the surface because just as H_5H_6 any of these edges is completed to a k^4 cutting l by a cubic.

Consequently the surface is of the degree nine; besides along l it cuts R_3 along a curve k^8 of the order eight, the locus of the point triplets which, together with the points of l, form quadruplets of I^4.

This ensues also from the way of producing I^4 of § 7. If P_1 describes the line l, the plane π_1 through the three points corresponding to P_1 in I^4, describes a pencil which has as axis the line m associated to l relative to ω^2. Any plane π_1 of this pencil contains three points outside m of the curve associated to l, i.e. the points of intersection of π_1 with the curve through P_1 of one of the six congruences of REYE found in § 7. As the curves of these congruences which cut l form a surface of the fifth degree, the curve corresponding to l in I^4 has five points in common with m, which, therefore, is indeed a curve of the order eight, k^8. The line l is a chord of k^8. For this curve cuts l in the two points of intersection of l and ω^2, because one of the points associated to such a point in I^4, coincides with this point of intersection. In each of the remaining fourteen points of intersection of k^8 with ω^2 there coincide two points associated to a point of l.

The projection k'^8 of k^8 out of a point of m on an arbitrary planeis

a curve of the order eight with a five-fold point M, through which point there pass fourteen tangents. If k'^8 had no multiple points besides M, this number of tangents would be 26. Hence the curve k'^8 has six more double points and it is, therefore, of the genus five. Through a point of m there pass six chords of k^8.

One of the three points which in I^4 are associated to the points of intersection of l with a plane δ of the configuration $\triangle$, lies in a point of $\triangle$, to wit in the pole of this plane relative to ω^2. The other two with $l\delta$ form a triplet of an involution of REYE in δ; k^8 cuts δ in these points and in the six points of $\triangle$ in δ.

Consequently the point triplets which in I^4 are associated to the points of a line l, form a curve of the order eight and the genus five, k^8, which cuts l twice, the line m, associated to l relative to ω^2, five times, and which passes through the points of the configuration $\triangle$.

The five points of intersection of k^8 with m are parts of quadruplets of I^4 of which one point belongs to l while the plane through this point and two of the other points of the same quadruplet, passes through l.

There are five quadruplets of I^4 which define tetrahedrons of which one side passes through a given straight line while one of the angular points in this side lies on the given line.

It appears also in the following way that through a point of m there pass six chords of k^8. A chord through a point P of m cuts k^8 in a pair of points which is a part of a quadruplet of I^4 of which one point P_1 lies on l and another point P_2 is a point of intersection outside l of k^8 with the polar plane π of P relative to ω^2, as $P_1 P_2$ must be associated to this chord relative to ω^2. π cuts the curve k^8 in six points outside l each of which, together with a point of l, forms a pair of points of I^4.

Consequently the locus of the pairs of I^4 in π cuts l six times. A point of intersection of π with any of the twenty straight lines of $\triangle$ together with the point in which the line associated to this line relative to ω^2 cuts π, forms a pair of I^4. Each of the points of the triplet of I^4 which belongs to π, together with the other two points of this triplet, forms two pairs of I^4.

The locus of the pairs of points of I^4 belonging to a given plane, is, therefore, a curve of the sixth order which cuts the twenty straight lines of $\triangle$ and which has double points in the three points of the triplet of I^4 lying in this plane.

§ 11. If, for instance, l cuts the line d_{123} of $\triangle$, d_{456} splits off from k^8 and there remains, accordingly, a curve k^7 of the order seven which passes through the twelve points of $\triangle$ outside d_{456} and which cuts m four times. Outside l this curve has twelve points in common with ω^2. Hence k^7 is of the genus four. In this way we may also examine the

curve that is associated to a straight line which cuts more than one line of $\triangle$.

Let us now consider the case that l passes, for instance, through the point D_{12} of $\triangle$. We use the way of producing I^4 of § 7 and that by the aid of a congruence of REYE which has a base point in D_{12}. Again a plane through m cuts the curve associated to l in three points outside m and this curve cuts m in the points of intersection of m and the quadratic cone which is formed by the curves of the congruence which cut l. To l there corresponds, therefore, a curve k^5 of the fifth order, which has l and m as chords. This curve passes through the points of $\triangle$ outside δ_{3456}, cuts ω^2 outside l in eight points and is, therefore, of the genus two.

In § 10 we found that there are fourteen quadruplets of I^4 which have one point on l and of which two other points coincide. Accordingly the surface of the bifurcation-points of I^4 is of the degree fourteen. We shall call it ω^{14}. The quadruplets of which one point lies in a given point of a straight line of $\triangle$, consist of two fixed points of the straight line and the pairs of an involution on the line associated to it relative to ω^2, so that among the said quadruplets there are two with two coinciding points. Consequently any point of a straight line of $\triangle$ is a double point of ω^{14}. As further, as appeared above, a straight line through a point of $\triangle$ contains eight bifurcation-points of I^4 besides this point, any point of $\triangle$ is a six-fold point of ω^{14}. A plane of $\triangle$ cuts ω^{14} along the four straight lines of $\triangle$ in this plane, each counted twice, and along the bifurcation-curve of the involution of REIJE which is associated to the pole of this plane relative to ω^2. This bifurcation-curve is of the sixth order and has a double point in each point of $\triangle$ in this plane.

Accordingly the locus of the bifurcation-points of I^4 is a surface of the fourteenth degree, ω^{14}, which has the straight lines of $\triangle$ as double lines and which has sixfold points in the points of $\triangle$.

§ 12. The ∞^3 carriers of the pairs of points of I^4 form a complex of which we shall determine the order by investigating which lines this complex has in common with the congruence of the chords of a twisted cubic k^3 of one of the congruences of REYE by the aid of which I^4 may be produced according to § 7. The common lines are in the first place the sextuplets of the rays joining any two points of a quadruplet of I^4 on k^3. Through any point of k^3 there pass three such lines so that a chord of k^3 cuts six generatrices of the surface formed by these rays, which surface is, therefore, of the sixth degree. As a point of $\triangle$ forms a pair of I^4 with any point of its polar plane relative to ω^2, all the points of $\triangle$ are cardinal points of the complex under consideration. This has, therefore, besides, five quadratic cones in common with the congruence of the chords of k^3; together with the found surface of the sixth degree, these form a scroll of the degree sixteen. There are no

other carriers of pairs of I^4 which are, at the same time, chords of k^3, because such a carrier would be a chord of two curves of the congruence of REYE of which k^3 is a part, hence a singular chord, and singular chords are only the lines through the base points of the congruence, which have already been counted. As the congruence of the chords of k^3 is a congruence (1,3), we find the order four for the complex of the carriers of the pairs of I^4. Any line of a plane of $\triangle$ as well as any line cutting two lines of $\triangle$ associated to each other relative to ω^2, is a carrier of a pair of I^4.

The carriers of the pairs of I^4 form a biquadratic complex C^4. The fifteen points of $\triangle$ are cardinal points, the fifteen planes of $\triangle$ cardinal planes of C^4. This complex contains the ten bilinear congruences each of which has as directrices a pair of lines of $\triangle$ associated to each other relative to ω^2.

The complex curve of C^4 belonging to a given plane α, has as tangents the fifteen lines of intersection of α with the cardinal planes of $\triangle$, the ten joins of the points of intersection of α with the pairs of directrices of the bilinear congruences belonging to C^4, and the three carriers of the pairs of the triplet of C^4 belonging to α. The complex cone of C^4 with vertex T has as generatrices the fifteen joins of T with the points of $\triangle$, the ten transversals through T of the pairs of directrices of the bilinear congruences belonging to C^4, and the lines which join T to the three points which, together with T, form a quadruplet of I^4.

§ 13. We shall now examine the surface which is the locus of the triplets which form quadruplets of I^4 with the points of a given plane φ. To each point of intersection of this surface with a given line l there corresponds in I^4 a point of φ lying in a point of intersection with φ of the curve k^8 associated to l. Hence the surface corresponding to φ in I^4 is of the eigth degree. We shall call it ω^8.

To the point of intersection of φ with a straight line of $\triangle$ there correspond the pairs of points of an involution on the line associated to it relative to ω^2 and one point of the same straight line. Hence the surface ω^8 contains all the lines of $\triangle$ and it has a double point in any point which is associated relative to ω^2 to the intersection of φ with a line of $\triangle$ and which belongs to the same line.

As to a straight line through a point of $\triangle$ there corresponds a curve of the fifth order, such a line has five points in common with ω^8 besides the said cardinal point of I^4, so that any point of $\triangle$ is a triple point of ω^8.

The surface ω^8 cuts a plane of $\triangle$ along the four lines of $\triangle$ in this plane and the quartic corresponding to the intersection of φ and the above mentioned cardinal plane of I^4 in the triple involution of REYE formed by the triplets of I^4 in this cardinal plane. The intersection of

ω^8 with φ consists of the curve of the sixth order c^6 which is the locus of pairs of points of I^4 in φ, and of the intersection of φ with ω^2.

Besides in the points of $\triangle$, ω^8 has a triple point in the pole F of φ relative to ω^2, as this point is associated to the three points of the triplet of I^4 which lies in φ.

A pair of points (P_3, P_4) which forms a quadruplet of I^4 together with a pair (P_1, P_2) of this involution which lies in φ and which, therefore, belongs to c^6, always lies on a straight line through F. Inversely to any pair (P_3, P_4) on a carrier through F there corresponds a pair (P_1, P_2) of c^6. The locus of the pairs (P_3, P_4) lies on the complex cone $\varkappa^4$ of C^4 with vertex in F. It has a triple point in F as F forms a pair (P_3, P_4) with any point of the triplet of I^4 in φ; it cuts any plane through F in eight points outside F and is, therefore, of the eleventh order. This curve c^{11} is a double curve of ω^8 as the points P_3 and P_4 are always associated in I^4 to each of the corresponding points P_1 and P_2 of φ.

The points where a transversal through F of the straight lines of $\triangle$ associated to each other relative to ω^2 cuts these lines, form a quadruplet of I^4 with the points of intersection of these lines and φ. For this reason c^{11} passes through the former two points. In accordance with this we found already that ω^8 has a double point outside the points of $\triangle$ on any line of $\triangle$.

The line which joins F to a point of $\triangle$, cuts the polar plane relative to ω^2 of the latter point in a point which forms a quadruplet of I^4 with a pair of points of the line of intersection of this polar plane with φ and the chosen point of $\triangle$. Consequently the curve c^{11} passes through all the points of $\triangle$ and through any point of intersection of the join of F and a point of $\triangle$ with the polar plane relative to ω^2 of the latter point.

Accordingly six of the points of intersection of c^{11} with a plane of $\triangle$ lie in the points of $\triangle$ in this plane, four lie on the straight lines of $\triangle$ in this plane and the eleventh point of intersection lies in the point where the plane is cut by the line which joins the pole relative to ω^2 of this plane to F.

The locus of the point triplets which form quadruplets of I^4 with the points of a plane φ, is a surface of the eighth degree which contains the lines of $\triangle$, has triple points in the points of $\triangle$ and in the pole F of φ relative to ω^2, and which has a double curve of the eleventh order.

§ 14. The double curve c^{11} of ω^8 cuts φ in the first place in the three points of the triplet of I^4 belonging to φ. In the other eight points of intersection a point P_2 of a pair of I^4 in φ must coincide with a point P_3 of the corresponding pair (P_3, P_4). These eight points of intersection lie, therefore, on the intersection c^2 of φ with ω^2. They also belong to c^6 as well as to c^{11}.

In the other four points of intersection of c^6 and c^2 the two points of a pair (P_1, P_2) in φ coincide. Hence the congruence K of the carriers of the coincidences among the pairs of I^4 has the class four. In the same way the two points of a pair (P_3, P_4) on a carrier through F coincide in the fourteen points of intersection of c^{11} with ω^2 which are different from the eight points already found. Accordingly the order of K is fourteen.

The carriers of the coincidences among the pairs of I^4 form a congruence $K(14,4)$.

The pairs of I^4 which belong to a straight line of $\triangle$, form an involution. As among these pairs there are two coincidences, any line of $\triangle$ is a double line of K. The same holds good e.g. for the line of intersection of the two planes δ_{1234} and δ_{1256}, as this line with $D_{13}D_{14}$ and $D_{15}D_{16}$ forms a degenerate curve of the congruence of REYE with $D_{12}, \ldots, D_{16}$ as base points, by the aid of which I^4 may be produced according to § 7.

Hence seven double rays of K pass through D_{12}, to wit the four lines of $\triangle$ through D_{12} and the lines $(\delta_{1234}, \delta_{1256})$, $(\delta_{1235}, \delta_{1246})$ and $(\delta_{1236}, \delta_{1245})$.

The planes of $\triangle$ are singular for K. The lines of K in a plane of $\triangle$, i. e. the bifurcation-lines of the involution of REYE in this plane of point triplets of I^4, envelop a curve of the sixth class. This curve has as double tangents the four lines of $\triangle$ in the plane and the six lines of intersection with the planes of $\triangle$ which have only two indices in common with the plane.

§ 15. The curve k^8 and the surface ω^8 corresponding resp. to the line l and the plane φ, have 64 points in common. Three of them lie in each of the fifteen points of $\triangle$. Among the other nineteen there are three which form a quadruplet of I^4 with the point $l\varphi$. There are eight points P_1 of l that belong to the same quadruplet of I^4 with another point P_2 of φ which always lies in a point of intersection of k^8 and φ. The eight pairs of points which complete these pairs (P_1, P_2) to quadruplets of I^4, give sixteen more points of intersection of k^8 and ω^8.

The surfaces ω^8 and ω'^8 corresponding to the planes φ and φ' have an intersection of the order 64. To this intersection belong the twenty lines of $\triangle$ and the curve k^8 which is associated to the line $\varphi\varphi'$. There remains a curve of the order 36, the locus of the pairs of points which complete a pair (P_1, P_2) of which P_1 lies in φ, P_2 in φ', to a quadruplet of I^4. This curve k^{36} has a quadruple point in each point of $\triangle$ as the corresponding triple involution of REYE contains four triplets of which one of the points lies in φ and another point in φ'. If d and d' are two lines of $\triangle$ associated relative to ω^2, φd, $\varphi' d'$, and the points of d and d' associated relative to ω^2 resp. to φd and $\varphi' d'$, form a quadruplet of I^4. In the same way a quadruplet of I^4 may be derived from the

points $\varphi' d$ and $\varphi d'$. Accordingly k^{36} cuts each straight line of $\triangle$ besides in three points of $\triangle$ in two more points which are associated relative to ω^2 to the points of intersection of this line with φ and φ'.

As k^{36} cuts a third plane φ'' in 36 points, we find:

There are 36 quadruplets of I^4 which have a point in each of three given planes.

The curve k^{36} cuts a third surface ω''^8 in $36 \times 8 - 15 \times 12 - 20 \times 2 = 68$ points which are not singular for I^4. Hence three surfaces ω^8, ω'^8, and ω''^8 corresponding resp. to the planes φ, φ', and φ'', have $68 + 19 = 87$ isolated points of intersection. Among them are the three points which form a quadruplet of I^4 with the point $\varphi \varphi' \varphi''$ and the 48 points of the $3 \times 8 = 24$ pairs which complete a point of the intersection of two of the planes φ, φ' and φ'' and a point of the third plane to a quadruplet of I^4. The remaining 36 points are those which form quadruplets of I^4 with three different points of which the first lies in φ, the second in φ', and the third in φ''.

Physics. — *"Measurements on the surface tension of liquid neon".* (Communication No. 182*b* from the Physical Laboratory, Leiden.) By A. TH. VAN URK, W. H. KEESOM and G. P. NIJHOFF.

(Communicated at the meeting of April 24, 1926).

§ 1. With the same apparatus, which was used for the measurement of the surface tension of liquid helium [1]), we determined the surface tension of liquid neon. The method used was just the same as in the case of helium. In order to get pure liquid neon in the reservoir with the two capillary tubes the apparatus was joined by means of a steel capillary to a big glass retainer formed by two glass bulbs (see fig. 1, A). These bulbs were previously exhausted and the vacuum controlled during some days. The neon was purified by freezing it in liquid hydrogen, after which the first part of it was pumped away, and only the middle part was used for the filling of bulb *B*, which then was blown to the apparatus.

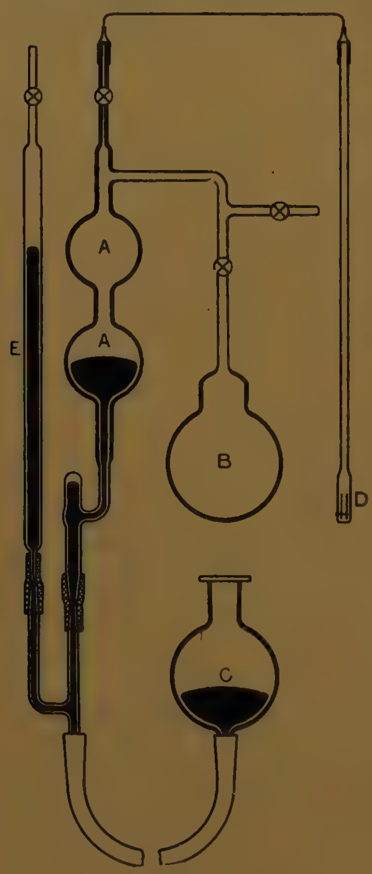

Fig. 1.

At the beginning of the measurements the gas was liquified in the reservoir *D*, which was held at the triple point temperature of neon, by raising the mercury reservoir *C*. For these measurements the reservoir *D* was put into the hydrogen vapour cryostat. [2]).

By regulating the height of the mercury reservoir *C*, we made certain that a large amount of liquid was first formed. Then some of it was allowed to evaporate in order to be sure that the walls of the capillary tubes were well wetted.

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[1]) A. TH. VAN URK, W. H. KEESOM and H. KAMERLINGH ONNES. These Proc. **28** 958, 1925; Comm. Leiden No. 179*a*.

[2]) These Proc. **19**, 1049, 1917 and **23**, 1185, 1921; Comm. Leiden Nos. 151*a* and 154*c*.

§ 2. The results of the measurements were calculated in the same way as given in the communication about helium. [1] They are given in the table. [2] In fig. 2 the values of ψ_M are plotted against T, the o and △ referring to the two capillaries placed alongside each other.

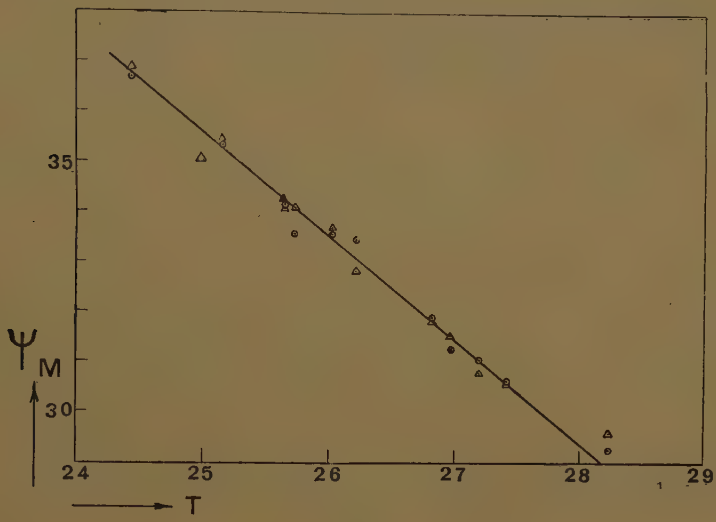

Fig. 2.

The constants for v. d. WAALS' formula

$$\psi_\sigma = A\ (1-t)^B$$

become $A = 14.73$ and $B = 1.202$.

The molecular surface tension is given by

$$\psi_M = 2.1\ (41.78 - T)$$

in which 2.1 is EÖTVÖS's constant.

By extrapolating down to $T = 0$ one gets for the surface tension at that temperature 1.18×10^{-14}.

For the heat of evaporation one finds at $T = 0$ if one extrapolates linearly the known values [3]) between 33° and 25° K.

$$\frac{20.2 \times 30 \times 4.19 \times 10^7}{6.06 \times 10^{23}} = 4.2 \times 10^{-14}$$

and so we find for the ratio $\frac{\sigma_0}{L_0}$ [4]) 0.28, a value in very good agreement with the calculations in Comm. Leiden, Suppl. N°. 54.

§ 3. *Measurements with oxygen, nitrogen, argon and hydrogen* have

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1) See note 1, p. 914.
2) The densities have been taken from Comm. Leiden, No. 162*b*.
3) Comm. Leiden, No. 162*b*.
4) These Proc. 28, 356, 1925; Comm. Leiden, Suppl. No. 54.

also been done by us. As the results of those measurements agree within the limits of accuracy with the existing measurements [1]), there is no need to give them here.

Surface tension of Neon.

T	First capillary			Second capillary		
	hr	Ψ_σ	Ψ_M	hr	Ψ_σ	Ψ_M
24.44	0.00939	5.75	36.7	0.00941	5.76	36.8
24.98	900	5.46	35.1	900	5.46	35.1
25.15	908	5.50	35.3	911	5.52	35.5
25.65	880	5.29	34.2	878	5.28	34.1
25.65	882	5.30	34.2	882	5.30	34.2
25.73	865	5.19	33.6	869	5.28	34.1
26.03	866	5.18	33.5	871	5.21	33.7
26.21	865	5.16	33.5	849	5.06	32.8
26.82	829	4.88	31.9	827	4.87	31.8
26.97	813	4.78	31.3	820	4.82	31.5
27.20	810	4.74	31.1	803	4.70	30.8
27.42	800	4.67	30.6	799	4.66	30.6
28.23	768	4.42	29.2	777	4.47	29.6

[1]) E. C. BALY and F. G. DONNAN, Journ. Chem. Soc. London, Transactions **81**, 907, 1902. H. KAMERLINGH ONNES and H. A. KUYPERS, These Proc. **17**, 525, 1914; Comm. Leiden No. 142*d*. See for the results J. E. VERSCHAFFELT, Mededeeling uit het Nat. Lab. Gent, No. 2, 1925.

Botany. — *"Vegetative Cultivation of Hippeastrum."* By Miss IDA LUYTEN. First Part. (Communication N⁰. 20, Laboratory for Plant-physiological Research, Wageningen.) (Communicated by Prof. A. H. BLAAUW.)

(Communicated at the meeting of April 24, 1926).

§ 1. *Introductory.*

The bulbs of the *Hippeastrum*-hybrids (Amaryllidaceae) have already been offered to the trade for several decades; usually however they have been offered under the incorrect name of *"Amaryllis Hippeastrum"* or *"Amaryllis-hybrids"*.

The original species come from South-America. According to inquiries made the species of this genus were imported into Holland in the order of succession subjoined:

about	1700	Hippeastrum	equestre Herb.
	1728	„	Reginae Herb.
	1769	„	vittatum Herb.
	1777	„	reticulatum Herb.
	1810	„	rutilum (fulgidum Herb.)
	1814	„	psittacinum Herb.
	1819	„	aulicum Herb.
	1820	„	solandriflorum Herb.
	1863	„	procerum Lemaire.
	1867	„	pardinum Dombrain.
	1869	„	Leopoldi Dombrain.

According to BAKER [1]) (1888) the first hybrid was obtained in 1799 by a watchmaker JOHNSON of Prescot (Lancashire), who crossed the species *Reginae* with the species *vittatum*. It appeared that on crossing the broad-leaved species were easily to yield good seeds and that the hybrids first obtained were useful fur crossing. About 1830 the chief species usually crossed, were: *Reginae, reticulatum, vittatum, aulicum,* and *solandriflorum;* more rarely used were: *equestre, fulgidum* and *stylosum.*

After 1870 the species imported by the English firm VEITCH: *pardinum* and *Leopoldi* were frequently used for crossing. The former yielded the spotted types in the hybrids of VEITCH.

1) BAKER, J. G. Handbook of the Amaryllideae.

We were told by growers of much experience, that especially the import of *pardinum* (1867) made the hybrids much more beautiful, but according to their experience their flowering-ability was reduced. As to this we deem it not impossible, that the wealth of flowers may be improved by certain conditions. On this important point experiments will be made in our laboratory to find the best conditions for flowering-ability.

Accordingly these bulbs are propagated by crossing fine, selected varieties, the flowers of which excel in colour, size and position. From this seed bulbs may be obtained, flowering as a rule afters 3 or 4 years. This method of propagation has a disadvantage, viz. that it is uncertain, how the seedlings will flower, whether they will really meet the requirements put to the flowers. As all *seedlings* may vary slightly, but sometimes even greatly, "named" bulbs cannot be supplied. The buyer therefore can never get at the same time or a couple of years at a stretch a number of bulbs *exactly* identical to each other. Growers do sell named grown seedlings of a fine prize plant; these are the descendants of an individual prized at the time. They resemble the original, also in consequence of good selection, but they are not the same being as the former prize-generation. Virtually named specimens must not be supplied, though it is customary and hitherto the only practical method.

Now some years ago we put ourselves the question whether it would not be possible to propagate these Hippeastrum-bulbs *vegetatively*, for instance by regeneration of the bulbscales cut away, in the way it is done with *Hyacinthus* on a large scale. As far as could be ascertained, growers had not yet succeeded in propagating the Hippeastrum-bulbs in this way. Could this really be done, the fine varieties once obtained could be preserved, their number increased and the bulbs supplied "named".

On inquiry it appeared but recently, that various growers were still experimenting on vegetative reproduction and that owing to the slight results some had not expected much from scooping or crossing these bulbs.

The bulbs yield but rarely or most rarely new bulbs originated from axillary buds. Had these been more frequent, they might have been oftener used to increase the number of a variety. From an old catalogue of the year 1862 of E. H. KRELAGE and SON of Haarlem, it appears, that the number could but be increased by a couple from such vegetative shoots. Whether the bulbs will form these offsets is uncertain; we must abide passively.

In practice some people are of opinion, that such bulbs vegetatively reproduced, will never flower.

§ 2. *Arrangement of the experiments.*

As the reports we received about vegetative reproduction were not very hopeful and there was little chance of success, the research in the laboratory

was arranged in such a way, that small quantities would be treated according to various methods. Thus we could trace whether there was any indication of regeneration and find the direction in which the research should be continued.

For starting the experiments the month was chosen in which the bulbs are put dry in culture to undergo a so-called period of rest in a lower temperature. This was done in accordance with the scooping and cutting of the Hyacinth; these bulbs too are used to be scooped and crossed some time after the lifting.

After receiving the bulbs on November 20, we put them in a room of 17° for some 10 days in order to be dried; in consequence of the pot-culture they were still slightly moist.

Next 48 combinations of 5 different external conditions, viz. species of soil, moisture, depth of planting, light and temperature were chosen, in which the different parts of the bulbs were to be treated, viz:

Peatdust	moist	below		in 13°, 20° and 27°
„	„	atop	light	
„	„	„	dark	
„	dry	below		
„	„	atop	light	
„	„	„	dark	
River sand	moist	below		in 13°, 20° and 27°
„	„	atop	light	
„	„	„	dark	
„	dry	below		
„	„	atop	light	
„	„	„	dark	
Calcareous dunesoil	moist	below		in 13°, 20° and 27°
„ „	„	atop	light	
„ „	dry	below		
„ „	„	atop	light	

On *Nov. 29, 1922* we scooped 48 bulbs, i.e. with a scooping-knife we cut the disk from the bulb, next the bulb was cut lengthwise into halves, so that the section ran parallel with the rest of the leaf-blades. This halving was necessary, because otherwise the bulbs could not be peeled without injury, the scales being closed; and in contradistinction to the scooped hyacinths where the scales are left together, we use them here separately. Every half bulb was carefully taken to pieces, so that the least damage possible was done. All parts were used, while the offsets sometimes present, were cleared off to prevent future mistakes. Next on all scales and leaves, coming from one half of the bulb, a diagonal crucial incision was made on the upper- and under-side at some distance from the base. It might be that near or on this crucial incision regeneration would be more likely to occur than near the wound at the base. The parts of a half bulb with and

without crucial incisions were brought into the 48 combinations, so that in this way 96 combinations were formed. The number of possibilities was increased by giving a certain position to the scales and leaves when they were put on or dug into the ground. It might be that the way in which gravitation effected the parts of the bulb, had some influence.

For this purpose those parts which were to be put "below" and bore no crucial incision were planted upright, the top parts of leaves or scales showing just above ground. One half of those bearing a crucial incision were as to the combination "below" put the convex side upwards, the other half the concave side upwards; next they were covered with a layer of soil. In putting them above ground this system of putting the concave side of half the number upwards the other half downwards was likewise applied.

The different pots were brought to three greenhouses; resp. kept at a constant temperature of 13°, 20° and 27°, so that there was plenty of light for that combination that was intended to get light. Those pots in which the bulb-parts lay on top, but needed dark, were covered with black paper. As the moist combinations dried rapidly without a black covering, these pots were covered with a wet piece of white muslin, transmitting much light and preventing a too rapid evaporation.

Besides in 13°, 20o and 27° a bulb was added, cut in two halves, one half of which was put on dry dune-soil, plane of section downwards, whereas in the other half the planes of section were exposed to the air.

The bulbs were supplied by Messrs VAN TUBERGEN, named *"Hippeastrum"*-hybrids. Dark red varieties were selected, since in case the experiments might prove a success, we should prefer this colour for further researches.

§ 3. *The origin of the vegetative bulbs.*

After some weeks (Dec. 18, 1922) it appeared, that the scales exposed to light, formed chlorophyll, especially in 20°, but also in 27° and hardly any in 13°. There likewise occurred a strong anthocyanin-formation. Besides the fact that bulb-parts were coloured entirely red, the planes of section were also coloured red. The strongest anthocyaninformation occurred in 13°. It was a remarkable fact that in that case anthocyanin was sometimes formed about 3 mms. from the wound of the crucial incision, filling in the space between the crucial incisions. Something similar we saw in the longitudinal sections of the scales. The plane of section was white and at 1—2 mms. distance behind it a stripe of anthocyanin was found.

At the beginning of January 1923 many experiments were stopped, because so many bulb-parts had rotten away. This was so bad in 13°, that but two combinations could be preserved. In 20° and 27° the rotting was less serious. Besides it appeared that the combination kept in dry soil, led to total desiccation and decay of the scales. The unhealthy material

IDA LUYTEN: "VEGETATIVE CULTIVATION OF HIPPEASTRUM". (First Part).

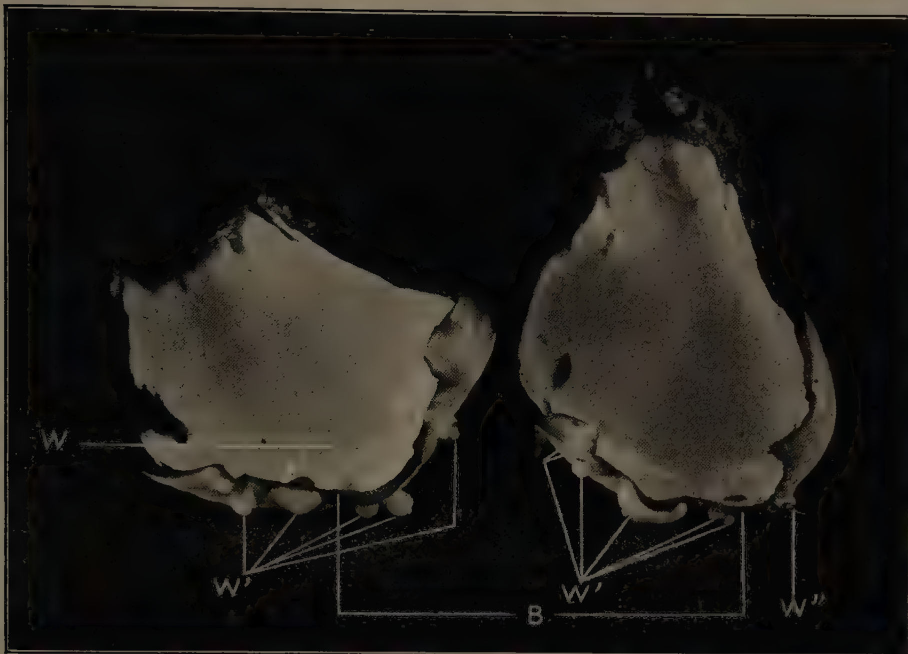

Fig. 1.

Fig. 2.

was cleared off, *all* the remaining combinations kept moist. Nothing however could still be observed, when here and there an individual was lifted.

On January 24, 1923, i.e. nearly 2 months after the experiments had been started, swellings were found along the planes of section of the bases of some scales in 27°. Fearing to injure something, we left the other experiments untouched and photographed only these scales. Fig. 1 shows the inside of a couple of half scales. Along the planes of section at the base (B) we notice smaller and larger protuberances (W, W', W''). We also noticed a scale bearing a small protuberance, already provided with a root. We point out that the scales after these two months look quite healthy. Further we notice that this regeneration may occur along the base at the inside of the scales (W), along the base at the outside (W'), and also that it need not arise close to the base, but may begin at some distance from the base (W'').

Now the question remained whether those protuberances and swellings would really produce healthy *Hippeastrum*-bulbs.

After a month (Febr. 23, 1923) little bulbs had grown from these tissue-growths, sometimes even with leaflets; they were potted up in the greenhouse in a soil-temperature of 25°—28° under glass cover and an air-temperature of 23°—26° C.

For an illustration of the young *Hippeastrum*-bulbs in a somewhat older stage than given in fig. 1, we refer to fig. 2. This is a photograph taken of some scales of *Hippeastrum*-bulbs, scooped a year later, Dec. 5, 1923. We are shown the lower surface of the bases of the scales.

In this photograph all kinds of forms may be found, viz. various stages of development and various modes of origin. 1 and 2 have already formed foliage-leaves (L), whilst 3 and 4 show a somewhat longer outer scale. From the opening of the scale at the top the first leaf will soon appear. Most of the little bulbs have not yet any roots; if the scales lie a little longer, the roots shoot either near the base of the bulbs or they appear later after the bulbs have been potted. N⁰. 5 has already formed a root (R). As stated above the place of origin of the tissue-growths may vary, but the primary development also appears to be different. Bulbs are namely found, grown from the first tissue-growth. In that case the base of the bulb is directly attached to the scale (1 and 3), others arise on a tissue-protuberance previously formed and stand more or less on a pedestal (5). N⁰. 6 apparently starts forming bulbs on the tissue-formations. At (7) various bulbs are developing on an older tissue-protuberance; a root *(R)* is already present. It sometimes happens, that the wound at the base of the scale gives a root (R'), without ever getting to regeneration of any other tissue on that spot. It may also occur, that instead of a fine, smooth, glossy-white tissue-growth, a more glassy rough-looking protuberance arises (Fig. 2 at 8). Such growths usually spread without yielding any bulbs. Yet they are sometimes found to have formed a bulb (2).

The above is a comparatively superficial description of what is noticed on examining the scales. A special research may trace the origin of this regeneration, i.e. the first cell-division, the further course of the origin of the various tissues and the various modes of formation of the bulbs. Next it may be accurately determined, at what distance from the base regeneration can occur. It may be pointed out already now, that it is not less than 3 mms. After the base had produced various bulbs, a strip of a scale, 3 mms. wide was cut off along the base. On this new wound new bulbs originated.

§ 4. *Result of the Experiments.*

When regenerative phenomena indicated, that experiments should be made in a comparatively high temperature, the pots from 13° were transmitted to 27°, the more so as the low temperature did not effect the scales favourably. On our continuing the experiments regeneration appeared to occur also at 20°, but much later and to a less degree. This latter is shown in table 1. Besides the bulbs took a longer time to attain a certain size.

TABLE 1. Number of bulbs on May 23, 1923.

	In 20°	In 27°	Light 20°	Dark 20°	Light 27°	Dark 27°	Without cruc. incisions 20°	With cruc. incisions 20°	Without cruc. incisions 27°	With cruc. incisions 27°
Peatdust	22	39	—	22	—	39	17	5	26	13
Riversand	20	36	4	16	22	14	14	6	21	15
Dunesoil	15	28	14	1	25	3	8	7	9	19
Total	57	103								

From table 1 various conclusions may be drawn, yet we should be careful in attaching too much value to the great difference in number of bulbs obtained in some experiments. The following items namely have influenced the result too much.

I. When the bulbs had reached a size of ± 8 mms., or even before that, when they bore a foliage-leaf very early, they were potted up. We did not risk separating these bulbs from the scale on which they had originated, because we did not yet know the strength of this new generation. Accordingly the young bulbs were potted up together with the whole scale or a large part of it. On account of this a great quantity of material which might have borne new bulbs, got lost. For our experiments in later years the bulbs were removed with a sharp knife together with a piece of scale as small as possible, so that the scale could continue yielding new bulbs.

II. In this first year we did not yet know in what temperature and what

Fig. 3. Vegetatively cultivated Hippeastrum after about one year.

Fig. 4. Vegetatively cultivated Hippeastrum after about two years.

combination the bulbs were which should give a small or a large crop. When later on the bulbs were numbered, it appeared, that between the individual bulbs there was a great difference in yield of young bulbs.

III. By keeping the material initially dry, much of it had become useless.

IV. The number of scales in 27° exceeded the number in 20°, because combinations from 13° had been transferred to 27° (see above).

V. The experiments were stopped on May 23, 1923, while even more bulbs could have been formed, as became evident in following years.

Yet the following conclusions may be drawn in combination with the observations made during the care of the experiments:

The higher temperatures, as stated above, had a more favourable influence than the lower.

Keeping the scales dry appeared to be injurious.

Light or dark appears to be of no importance. As with respect to keeping the scales moist digging them in a little is much more favourable, the latter was recorded as better treatment. Whether the concave or convex side faced downward has no noticeable influence on the origin of the young bulbs. Later experiences taught us, that practically it is best to put the scales in a slanting position, the concave side upwards.

Near or on the crucial incisions there never occurred any regeneration; neither had it any favourable influence on the origin of bulbs along the base. It was even a disadvantage, that in consequence of the crucial incision, the scales were sooner subject to rotting.

As to the choice of soil, neither did this give rise to great differences; we got the impression however that on account of its sourness the moist peatdust made the scales rot sooner.

Never were regenerative-phenomena met on the foliage-leaves, flower-stalks or the disk. These parts of the bulb therefore were removed on scooping in later years when the experiments were started. Potting up the disks may be worth while however. If namely the vegetation-point has remained intact, it may continue splitting and by doing so restore the old bulb. In our case 5 bulbs, treated in this way were preserved; 2 of them flowered again after 3 years.

It should be mentioned, that in the halved bulbs months later there appeared minute swellings at the base of the halved scales. They were so small that compared to the other results, this treatment evidently offered no advantages.

It will be investigated into whether another month than Nov.-Dec. is more favourable with regard to scooping.

§ 5. *Cultivation and bringing into bloom of the bulbs vegetatively obtained.*

Now it still had to be proved whether these little bulbs really had the

same vitality as the seedlings. In the previous § we already pointed out, that when the bulbs on the scales had attained a certain size, they were potted up. In the first year of the experiments the whole scale was potted up with them, or when each half bore a bulb, half the scale. If a couple of bulbs grew close together, they were put in the same pot. In later years it appeared that such bulbs could be separated. Even when they are growing on one protuberance, we may halve the pedestal without causing a lasting injury.

As soil we chose equal parts of sieved peatdust and dunesoil, the pots being dug in moist peatdust in a plant-box in the greenhouse. The temperature in this greenhouse amounted to 21°—23°, while the temperature of the soil in the plant-box was 25°—28°. To keep the space moist the plant-box was covered with glass, which kept the temperature under the glass at 23°—26°. To the right of figure 3 we see part of such a closed box. When the plants had struck root and become vigorous, they were removed to a box with the same temperature of soil, but without glass-cover.

Part of the material of the 27° experiments was potted up on *Febr.* 23, 1923 and consisted of 26 individuals. From 20° the first 8 individuals were put under glass on *March* 5. The harvest continued till May 23, the point of time, when the experiments were stopped in the first year. In later years the experiments were continued much longer.

On that date 103 bulbs had been obtained in 27°, 57 in 20°. When such young plants have attained an age of 7—10 month, they look as represented in fig. 3. On Oct. 23, 1923, the bulbs were put in larger pots if necessary, and the soil was made a little heavier with clay. The bulbs were kept growing.

Fig. 4 gives the size of the plants, when they are a year older again (Nov. 30, 1924). On the previous Sept. 27, they had again been put in larger pots and the soil had been made heavier. So they passed the winter and summer of 1925.

The vigour of the plants had already proved, that bulbs vegetatively obtained, remain healthy and develop as well as seedlings.

Now it had to be proved whether the bulbs would really yield flowers, which was doubted by some people. For this purpose all plants were put dry at the beginning of Oct. 1925 and in the middle of October (the leaves are decaying by that time) transmitted to a lower temperature (16°). On Dec. 17 the first inflorescence shows in a bulb potted up on April 6, 1923 as scale-bulb from a combination 27°, dunesoil. On Dec. 30 the bulb was put in ± 23° and kept moist, so that the first flower opens on Jan. 23, 1926. Fig. 5 is an illustration of this first flowering *Hippeastrum* vegetatively obtained. At the bottom near the bulb a second inflorescence may be discovered (B 2).

That these plants equal the seedlings in strength follows from this figure, but is moreover proved by fig. 6 (photographed Febr. 4, 1926), where the two vigorous stems simultaneously bear flowers. One stem

Fig. 5. The first flowering Hippeastrum cultivated from scales.

bears three, the other 4 flowers. This bulb obtained from a bulb scooped on Nov. 29, 1922 was potted up on April 6, 1923 and flowers ± Febr. 4 1926. Within three years it has grown into a flowering plant.

It might be expected that not all bulbs should flower in 1926, no more as is the case with the seedlings. In the course of the 3 years 6 bulbs got lost, three of which died. 154 bulbs were left, which may be classed as follows :

TABLE 2. Flowering after ± 3 years.

<table>
<tr><th rowspan="2"></th><th rowspan="2">Peat dust 27°</th><th rowspan="2">River-sand 27°</th><th rowspan="2">Dune-soil 27°</th><th rowspan="2">Peat dust 20°</th><th rowspan="2">River-sand 20°</th><th rowspan="2">Dune-soil 20°</th><th colspan="2">Total</th></tr>
<tr><th>in 27°</th><th>in 20°</th></tr>
<tr><td>2 flowerstalks</td><td>6</td><td>4</td><td>5</td><td>—</td><td>—</td><td>1</td><td rowspan="2">43</td><td rowspan="2">6</td></tr>
<tr><td>1 flowerstalk</td><td>17</td><td>3</td><td>8</td><td>1</td><td>1</td><td>3</td></tr>
<tr><td>No flowerstalk</td><td>15</td><td>25</td><td>14</td><td>23</td><td>19</td><td>9</td><td>54</td><td>51</td></tr>
</table>

We read from this table, that nearly half the bulbs, originally formed in 27°, flowered within three years ; viz. 43 of the 97 individuals flowered, 15 of them with 2 flower-stalks. From the bulbs obtained in 20° only 6 of the 57 individuals flowered. We may conclude with certainty, *that the bulbs from 27° are more vigorous, than the bulbs from 20°.*

As in the case of seedlings no more than 40 to 50 % flowers after 3 years, we may call the ratio obtained from 27°, really favourable, especially as of the various treatments applied some did not prove favourable, so that the cultivation of many weak bulbs was also continued. Later experiments will have to prove whether this percentage of bulbs producing flowers after 3 years cannot be increased, when namely after scooping the correct treatment is directly applied. Besides the continuation of our *Hippeastrum*-culture will prove whether the yield of flowers increases in the fourth year, which may surely be expected.

From the above it appears, that *in future* the grower can cultivate the *Hippeastrum pure in shape and colour,* i.e. as varieties — he may therefore in future supply the *Hippeastrum-bulbs named.* He may continue trying to gain new varieties from seed by crossing, while moreover he will be able to preserve pure varieties and to increase their number by propagation from scales.

Meanwhile the experiments have been continued every year since 1922. Since the first experiments indicated 27° as most favourable temperature and it proved best to put the scales in moist, not too sour soil, these results were used as a base. The young scale-bulbs, obtained from the same bulb,

were kept together under the same number, so that already now we possess lots of pure vegetative descent.

As soon as they flowered, we shall publish further data on the best mode of cultivation to our experience in a second paper. We can state already now, that 29°—30° C. and seed-pans with *glass cover* yield the most favourable results.

Our grateful acknowledgement is due to the *Hollandsche Cultuurmaatschappij* for their quite disinterested support in the extra-expenses of this and other publications of our laboratory.

We want to express our thanks to Messrs: C. G. VAN TUBERGEN of *Haarlem*, Messrs: DEN OLDER BROS, of *Leiden*, E. H. KRELAGE of *Haarlem* and Messrs: WARMENHOVEN of *Hillegom*, for the various informations on *Hippeastrum*-culture we might receive.

Laboratorium voor Plantenphysiologisch Onderzoek,
Wageningen.

February 1926.

EXPLANATION OF THE FIGURES.

Fig. 1 and Fig. 2 see text.

Fig. 3. Scooped on Dec. 4, 1923; the first potted up on Jan. 23, 1924, the last on Oct. 21. 1924. Photo Nov. 30, 1924.

Fig. 4. Scooped on Nov. 29, 1922; potted up from Febr. to the close of May 1922. Photo Nov. 30, 1924, i.e, a year older than in fig. 3.

Fig. 5. First flowering Hippeastrum cultivated from scales. Scooped Nov. 29, 1922, potted April 6, 1923; flowering Jan. 23, 1925.

Fig. 6. Hippeastrum cultivated from scales, 2 stems flowering simultaneously.

IDA LUYTEN: "VEGETATIVE CULTIVATION OF HIPPEASTRUM".
(First Part).

Fig. 6. Hippeastrum cultivated from scales, flowering with two stalks at the same time.

Neurology. — *"On Recovery of Function of the Facial Muscles after Hypoglossus-facialisanastomosis."* By Prof. E. D. WIERSMA.

(Communicated at the meeting of April 24, 1926).

A boy of 4 years was run over by a motorcar in August 1924. It is not quite sure whether he lost consciousness. An hour after the accident he vomited repeatedly. Directly after the trauma complete paralysis of the right n. facialis was established. The right corner of the mouth did not move when the boy was crying and the right eye could not be shut. The doctor who saw him an hour later stated deafness of the right ear. No further anomalies were found. The other cranial nerves and the extremities functioned normally. When in November 1924 the facialis-paralysis had not taken a turn for the better the patient was admitted to the neurological clinic of Groningen. There the verdict was given: complete deafness of the right ear and complete facialis-paralysis with reaction of degeneration on the right. For the rest there were no anomalies. After consultation with the surgeon a nerve-plastic was decided on. For the facialis-anastomosis only two nerves can receive consideration, viz. the hypoglossus and the accessorius. Although our considerations for choosing the hypoglossus are more appropriate to a surgical and neurological exposition, it is perhaps well to point out that, according to the experience of F. KÖNIG, E. LEXER and L. WREDE [1]) the hypoglossus-anastomosis is preferable to the accessorius-anastomosis, as well with regard to the secondary paralyses, as to the associated movements. My object in writing this article is to give an insight into the way in which the recovery of function of the paralyzed right half of the face was effected after the plastic. The hypoglossus was cut through, and the central end was attached to the peripheral end of the facialis, which also had been cut through. Before the cutting it had been ascertained that the exposed nerve did not respond anymore to electric stimuli. This operation was performed in the latter part of November. Of course it occasioned a complete paralysis of the right half of the tongue, which was soon followed by atrophy and reaction of degeneration.

This status persisted some time. The facialis muscles remained unchanged. Not before May 1925 did some spontaneous twitches of the right eyelids appear, which on closer examination had to be considered as associated movements. Whenever the tongue was moved while the boy was speaking or eating, contractions were perceivable of the right half of the face. At first they were still faint, but gradually they increased in

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[1]) BIER, BRAUN, KÜMMELL, Chirurgische Operationslehre. 1917. p. 456—459.

intensity and extent, while after some weeks the whole right half of the face was moving continually, when the boy ate or spoke. These associated movements were so strong that at times we doubted whether the operation had yielded any profit.

Other concomitant movements revealed themselves about the same time. Whenever the left eye was shut, the right eye was also closed; when the boy was laughing or weeping the right corner of the mouth moved together with the left one. These associated movements were hardly visible at first, but grew stronger every week. In this way a simultaneous cooperation of the two halves of the face was brought about.

Little by little the associated movements were further modified, for in January 1926 it became clear that when the patient was speaking or eating i.e. together with the movements of the tongue, the associated movements of the facial muscles were much weaker. The tongue could be put out without large contractions of the facial muscles making their appearance on the right side, and at length hardly any contractions were visible. In addition weak, voluntary movements could be made with the right corner of the mouth. This condition persisted up to now (April 1926). These associated movements recur only when the child gets very excited: when he gets angry or also when he is merry.

These three phenomena, i.e. the movements of the facial muscles on the right, accompanying the movements of the tongue, the recurrence of the simultaneous movement of either half of the face and the slow disappearance of the associated movements, accompanying the movements of the tongue (so that an increased dissociation of the movement of the tongue and the face is brought about as in normal cases) and the voluntary contraction of the right corner of the mouth, require further explanation.

Is is easy to see that after regeneration of the hypoglossus that is attached to the facialis, the facialis-muscles will contract at every impulse originating from the hypoglossus-centrum. This will also be the case with movement of the non-paralyzed half of the tongue, because the two halves of the tongue will always cooperate. The action of the one hypoglossus-centrum will of necessity activate the other.

It is not so easy to explain the recurrence of the simultaneous cooperation of the facial muscles on the left and on the right side. As a rule the two halves cooperate. There is a regular association between the motor impulses on the right and on the left. But in our case the impulses for the left half of the face cannot excite simultaneous movements for the right half, because the nerve that is to transmit those impulses to the right half, is paralyzed. If in spite of this there is really cooperation between the two halves it must be accomplished along other paths. To render this conceivable I point to the marked association that always exists between

movements of the tongue and those of the face. When the tongue is put out, the mouth-aperture is widened, the naso-labial folds are getting more conspicuous, sometimes the forehead is wrinkled, frequently the eyes are closed or opened wide. The converse association also exists. Contractions of the muscles of the face are accompanied by movements of the tongue. When the eyes are fast shut, when the forehead is extremely wrinkled, or when the mouth is forcibly widened, movements of the tongue can distinctly be observed in the open mouth. This is noticeable above all in children, and in grown-ups with a lively face-play. Our knowledge of these associated relations affords an explanation of the cooperation of the two halves of the face of our little patient. The motorial impulses, arising from the right facialis-centrum, and moving the left half of the face activate the right hypoglossus-centrum, by which, as stated above, also the activity of the left one is brought about. The impulses that arise here, will produce a contraction of the right facial muscles. This then explains that a bilateral cooperation of the muscles of the face occurs. This instance shows that under some conditions the neurologist can turn to account psychological ideas. If it were not known that anatomical connections exist between the hypoglossus-centra inter se, and between the facialis- and the hypoglossus-centra, the above psychological speculations would entitle us to say with certainty that they must exist and that an elaborate experimental procedure will most probably bring them to light.

The third phenomenon, viz. the slow disappearance of the associated movements of the facial muscles accompanying the movements of the tongue, and the voluntary contraction of the right half of the face, also requires further explanation. A dissociation occurs, the contractions of the face become more dissociated from the movement of the tongue. The impulses from the right hypoglossus-centrum are utilized for the muscles of the face.

Analogous phenomena occur in the recovery of function, which is observed in some physiological experiments.

KENNEDY attached the nerve that innervates the extensors in a dog's foreleg to the nerve supplying the flexors, and the reverse. After regeneration recovery of function was effected by some exercise. When after trepanation the extensor-centrum in the cortex was stimulated the leg was flexed, and the reverse. Here, then, a transformation of the function of the centra takes place, for the animal has learned by practice that an impulse towards the extensors is needed for flexion of the leg. This is quite in keeping with our case; the child has learned to move the muscles of the face by a stimulus towards the tongue.

OSBORNE and KELVINGTON [1]) have repeated these experiments in a slightly different way. After severing one of the main cord of the left plexus brachialis in a dog they attached the central end to the distal end

[1]) W. A. OSBORNE and BASIL KELVINGTON. Journal of Physiology, 1908

of the corresponding severed cord of the right plexus with due precaution to render only regeneration of the attached cords possible. After regeneration the animal learned by practising to use his foreleg again. When after trepanation the right hemisphere was stimulated the muscles of the left leg, which were innervated by the intact cords of the left plexus and also a movement of those muscles of the right leg which were innervated by the transplanted cord of the left plexus, could be seen. This goes to show that those movements are induced in the right leg by a motorial impulse towards the left leg. The dog has gradually learned this from experience, by observing sensations of feeling and here also by eye-sensation, just as in our case exercise has brought about the transformation of the hypoglossus-centrum.

The results of these physiological experiments and those of hypoglossus-anastomosis are to a great degree indicative of the development of the voluntary movements of the child. The neonatus shows only spontaneous movements and reflex-movements, from which voluntary movements gradually develop. The child is aware of every muscle-sensation, and these experiences will vary with the difference in duration, intensity and extent of the movement. Of every sensitive impression some trace is left behind, which belongs to a definite impulse. By way of those sensations the child will learn to find through exertion the right impulse for a wished for movement.

The great influence of sensitive impressions upon movements appears first of all from the fact that embryologically the sensitive paths are sooner developed than the motor ones, and secondly from the excellent comparative anatomical research by ARIENS KAPPERS [1]), which proves that the location of the sensitive paths and nuclei is constant, whereas that of the motor ones is variable and determined by the corresponding sensitive paths. Furthermore we must assume that an association appears between the sensations of the contraction of cooperating muscles and those of symmetrical groups of muscles. This enables us to imitate at will the attitude of the arm or leg on the opposite side. In the same way when the muscles of the face are in function a contraction will arise of equal intensity on both sides.

Just as in the child the voluntary movements will originate from the spontaneous movements and the reflex-movements, the animals of the above physiological experiments will learn through exercise and apprehension to alter undesigned movements into intentional movements. Our little patient thus apprehends that for every movement of the face a definite impulse starting from the hypoglossus-centrum is required. By dint of exercise harmony is established in the movement of the two halves of the face.

There is still another similarity with the development of voluntary

[1]) ARIËNS KAPPERS. Folia Neurobiologica. Bnd. I.

movements. The training for new intentional movements is first accompanied by a number of superfluous associated movements, e. g. for cycling, writing etc. These associated movements are after some time inhibited by the concentration of the energy upon the voluntary movements. They are not annulled, however, since they will recur when the said concentration is abated in one way or other. This becomes evident from the awkardness in emotional conditions, such as fear or anger, or from atactic writing, and from the accompanying movements of lips and face of some demented patients. In the same way the associated movements of the muscles of the face are in our patients not annulled but only inhibited, for they will recur directly with full vigour when the boy gets excited or angry or merry.

Now, how do the sensations of feeling arise that direct the motor impulses and are on that account of great significance for the voluntary movements?

Signal data on this point have been afforded by the very minute histological investigation by J. BOEKE [1]). After SHERRINGTON had demonstrated already some years ago that severing of the motor-nerves of the eye-muscles resulted in degeneration of sensitive end-bodies in the tendons, BOEKE established that the trochlearis and the abducens possess also centripetal fibres. Severing of these nerves brought about a degeneration of the sensitive end-bodies in the muscles and in the tendons. In the loose connective tissue in and round about the muscle a few sensitive end-bodies could still be found, which do not degenerate after the nerves of the eye-muscles are cut through, and whose impulses are probably directed to the centrum along the trigeminus. A similar condition was found by BOEKE for the hypoglossus. Most likely that of the facialis, with which we have to do here, does not differ much from it. Furthermore it is known that also in the joints sensitive nerves are lodged, which are stimulated when active or passive movements take place, and that the skin round about a point is more or less strained during a movement by which also stimuli run to the centrum. We are, therefore, safe to assume that sensation of movement can originate in different ways, viz. along the motor nerves and the sensitive nerves of the muscle and along the sensitive nerves of skin and joints. Also visual impressions can promote an accurate cooperation of the muscles. So there is a complex of causes for tactile sensation which is hard to disentangle. Most probably there are associations between those sensitive sensations, so that the one can compensate the other.

Little is known of each of these sensations. It would seem to me, that the clinic is competent to throw more light upon it. Sensation of movement can be established e.g. by moving the extremities in the joints passively and by recording the slighted movement observable. Another method

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[1]) J. BOEKE. Studien zur Nervenregeneration. Verhandelingen der Kon. Akad. v. Wetensch. Deel XIX, N⁰. 5.

might be employed which has been suggested by CURSCHMANN [1]). First the threshold value of the cathodal closure contraction is determined in m.A: and subsequently the threshold value at which the contraction is observed. These two values are under normal conditions nearly equal. In the same way the threshold value of the movement and of the sensation of it in the joints can be estimated.

A number of experiments on peripheral facialisparalysis showed me that the sensation of contraction appears a little later here than the minimal movement. It is probable, therefore, that the facialis conducts also centripetal paths.

However, the awareness of movement is also experienced along the trigeminus. This view is corroborated by a case in which the ganglion Gasseri was extirpated because of violent trigeminus neuralgia. This brought on complete insensibility of the right half of the face and of the right half of the tongue. On examining the muscular sense there appear to be considerable disturbances. On showing the teeth the right half of the face is moved less than the left half. It is evident that the harmony between the movement of the muscles of the face on the right and on the left is diminished. This is also borne out by the following experiment. When I ask patient to raise a little first the left corner of the mouth and then do the same with the right corner, the latter movement is much stronger. Are we perhaps to ascribe this to the fact that the same muscular sense is aroused on the left, only with a much stronger contraction ? On applying galvanic stimuli it also appeared that the muscular contraction was felt on the right only when it was much stronger than on the left. Likewise on the right only a stronger electric contraction of the tongue is observed, whereas on the left a very weak, hardly perceptible movement is already apprehended.

[1]) CURSCHMANN. Neurol. Centralblatt, 1915.

Mathematics. — "*On* RIEMANN*ian Geometries admitting an absolute parallelism.*" By E. CARTAN and J. A. SCHOUTEN. (Communicated by Prof. JAN DE VRIES).

(Communicated at the meeting of April 24, 1926).

We will say that a RIEMANNian geometry admits an absolute parallelism, if it is possible to define the parallelism of two directions in two different points in a manner, which is absolute (viz independent of the choice of the coordinates) and satisfies the following conditions:

1. A geodesic is in all her points self-parallel;

2. The angle between two different directions in an arbitrary point P is equal to the angle between the two parallel directions in another arbitrary point Q.

By the thus defined parallelism a connexion arises, which

1. leaves invariant the tensor $g_{\lambda\mu}$

2. possesses the same geodesics as the given RIEMANNian connexion

3. has zero curvature.

This new connexion is not necessary symmetrical. In such a RIEMANNian geometry evidently through every non-singular point not situated on a given geodesic one and only one geodesic may be drawn, which is in each of her points parallel to the given one.

In a previous paper we proved, that with every simple or semi-simple group such a RIEMANNian geometry corresponds and that the geometries corresponding with simple groups admit two different absolute parallelisms. The most simple case is the geometry of the elliptical S_3, the two parallelisms being those of CLIFFORD.

We will prove presently that, supposing the fundamental form definite, there exists besides these geometries corresponding with the mentioned groups only one other geometry with the designed property and that this geometry is in close connexion with the non-associative number-system of GRAVES—CAYLEY.

§ 1. *Fundamental relations.*

A connexion with the same geodesics as the given RIEMANNian geometry, has parameters of the form

$$\overline{\Gamma}^{\nu}_{\lambda\mu} = \overset{0}{\Gamma}{}^{\nu}_{\lambda\mu} + p_{\lambda} A^{\nu}_{\mu} + p_{\mu} A^{\nu}_{\lambda} + S_{\lambda\mu}^{\cdot\cdot\nu} \quad \ldots \ldots \quad (1)$$

where p_{λ} is an arbitrary vector and $S_{\lambda\mu}^{\cdot\cdot\nu}$ an arbitrary in $\lambda\mu$ alternating affinor and $\overset{0}{\Gamma}{}^{\nu}_{\lambda\mu}$ are the parameters of the RIEMANNian geometry.

If we postulate

$$\overline{\nabla}_{\omega}\, g_{\lambda\mu} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

it follows immediately

$$p_{\lambda} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$S_{\lambda\mu\nu} = S_{[\lambda\mu\nu]} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$S_{\lambda\mu\nu}$ is therefore a *trivector*. For the quantity of curvature of the new connexion it follows from (1)

$$0 = \overline{R}_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} = K_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} + 2\,\nabla_{[\omega} S_{\mu]\,\lambda}^{\cdot\cdot\nu} - 2\, S_{\alpha\,[\omega}^{\cdot\cdot\nu} S_{\mu]\,\lambda}^{\cdot\cdot\alpha} \quad . \quad . \quad (5)$$

and from (5) after some calculations

$$K_{\omega\mu\lambda\nu} = {}^1/_3\, S_{\alpha\mu\lambda}\, S^{\alpha}_{\cdot\nu\omega} - {}^2/_3\, S_{\alpha\lambda\nu}\, S^{\alpha}_{\cdot\mu\omega} + {}^1/_3\, S_{\alpha\nu\mu}\, S^{\alpha}_{\cdot\lambda\omega} \quad . \quad . \quad . \quad (6)$$

$$S_{\omega\mu\lambda\nu} = \overset{0}{\nabla}_{\omega} S_{\mu\lambda\nu} = {}^1/_3\, S_{\alpha\mu\lambda}\, S^{\alpha}_{\cdot\nu\omega} + {}^1/_3\, S_{\alpha\lambda\nu}\, S^{\alpha}_{\cdot\mu\omega} + {}^1/_3\, S_{\alpha\omega\mu}\, S^{\alpha}_{\cdot\lambda\mu} \quad . \quad (7)$$

From this latter equation it follows that $S_{\omega\mu\lambda\nu}$ is a *quadrivector*. By differentiation we deduce from (6) and (7), using the identity

$$\overset{0}{\nabla}_{\nu} K_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} = 2\,\nabla_{[\omega} K_{\mu]\lambda} \text{ [1]} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

the equation

$$\overset{0}{\nabla}_{\omega} K_{\mu\lambda} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

§ 2. *The case of constant real p-directions.*

If a real p-direction is constant by the (0)-connexion, then the orthogonal $(n-p)$-direction is constant also and it is well-known that V_n is composed by ∞^{n-p} V_p which are totally geodesic and mutually (0)-parallel and also in ∞^p V_{n-p} with the same properties orthogonal to the V_p. The trivector is the sum of two trivectors $S'_{\lambda\mu\nu}$ and $S''_{\lambda\mu\nu}$, which are entirely situated in the V_p and V_{n-p} resp. and the transformation (1) is composed by a geodesic transformation (viz. a transformation which leaves the geodesics invariant) of every V_p in itself, given by $S'_{\lambda\mu\nu}$ and an analogous transformation of every V_{n-p} in itself, given by $S''_{\lambda\mu\nu}$. Indeed, $K_{\omega\mu\lambda\nu}$ is divided, as is well-known, into two parts, which are situated entirely in V_p and V_{n-p} resp. If therefore v^{ν} and w^{ν} are two vectors lying entirely in V_p and V_{n-p} resp., then from (6) it follows:

$$0 = v^{\omega}\, w^{\mu}\, v^{\lambda}\, w^{\nu}\, K_{\omega\mu\lambda\nu} = w^{\mu}\, v^{\lambda}\, S_{\alpha\mu\lambda}\, w^{\nu}\, v^{\omega}\, S_{\nu\omega}^{\cdot\cdot\alpha} \quad . \quad . \quad . \quad (10)$$

The real vector $w^{\mu} v^{\lambda} S_{\alpha\mu\lambda}$ is therefore zero, whence it follows that $S_{\lambda\mu\nu}$ is divided in the described manner.

[1]) R. K. p. 168.

In consequence of this proposition the case, where V_p admits constant real p-directions is reduced to the case, where these p-directions do not exist. We will therefore suppose in the following, that there exist no constant real p-directions.

§ 3. $K_{\mu\lambda} = cg_{\mu\lambda}$.

From this supposition it follows that $K_{\mu\lambda}$ must be equal to $g_{\mu\lambda}$ but for a constant factor. Indeed, if this were not true, the principal regions of $K_{\mu\lambda}$ would define constant p-directions in consequence of (9). The case $K_{\mu\lambda} = 0$ is to be excluded immediately, for in consequence of (6) we would have $S_{\alpha\beta\gamma}S^{\alpha\beta\gamma} = 0$ which is by a real trivector only possible for $S_{\lambda\mu\nu} = 0$, giving the trivial case $K_{\omega\mu\lambda\nu} = 0$. We have therefore

$$K_{\mu\lambda} = cg_{\mu\lambda} \;\; ; \quad c = \text{constant} \neq 0. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

or

$$S_{\alpha\beta\lambda} S^{\alpha\beta}_{\cdot\,\cdot\,\mu} = - cg_{\lambda\mu}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (12)$$

§ 4. BIANCHI's *identity*.

Applying BIANCHI's identity on (6), we obtain

$$0 = {}^9/_4 \nabla_{[\xi} K_{\omega\mu]\lambda\nu} = S_{\beta\alpha[\mu} S^{\alpha}_{\cdot\,\omega\xi]} S^{\beta}_{\cdot\,\lambda\nu} + S_{\beta\alpha\lambda} S^{\alpha}_{\cdot\,\nu[\omega} S^{\beta}_{\cdot\,\xi\mu]} - S_{\beta\alpha\nu} S^{\alpha}_{\cdot\,\lambda[\omega} S^{\beta}_{\cdot\,\xi\mu]} \qquad (13)$$

From $S_{\lambda\mu\nu}$ we derive the covariants

$$\left.\begin{aligned} &a) \quad cg_{\lambda\mu} = S_{\alpha\lambda}^{\cdot\,\cdot\,\beta} S_{\beta\mu}^{\cdot\,\cdot\,\alpha}, \\ &b) \quad g_{\lambda\mu\nu} = S_{\alpha\lambda}^{\cdot\,\cdot\,\beta} S_{\beta\mu}^{\cdot\,\cdot\,\gamma} S_{\gamma\nu}^{\cdot\,\cdot\,\alpha} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (14)$$

etc.

and we remark that all these covariants admit cyclical permutation of the indices. From (13) it follows

$$c\, S_{\lambda\alpha[\mu} S^{\alpha}_{\cdot\,\omega\xi]} = - 2\, g_{\lambda\alpha[\mu} S^{\alpha}_{\cdot\,\omega\xi]} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (15)$$

By transvection of this equation with $S^{\xi\lambda\gamma}$ and $S^{\gamma\omega\xi}$ arises

$$2\, cg_{\mu\gamma\omega} - c^2 S_{\mu\gamma\omega} = 4 g_{[\dot{\omega}\dot{\gamma}|\dot{\delta}|\dot{\mu}]}{}^{\delta} + 2 g_{\gamma\alpha\beta} S^{\alpha\beta\gamma} S_{\gamma\mu\omega} \quad . \quad . \quad . \quad . \qquad (16)$$

and

$$2\, cg_{\mu\gamma\omega} - c^2 S_{\mu\gamma\omega} = 4\, g_{[\dot{\omega}\dot{\gamma}|\dot{\delta}|\dot{\mu}}{}^{\delta} + 2\, cg_{\mu\gamma\omega} \quad . \quad . \quad . \quad . \qquad (17)$$

hence

$$g_{\gamma\alpha\beta} S^{\alpha\beta\gamma} S_{\gamma\mu\omega} = - cg_{\gamma\mu\omega} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (18)$$

Now it follows from (14*b*) by differentiation

$$\nabla_{\omega}\, g_{\lambda\mu\nu} = - \frac{c}{2} \nabla_{\omega} S_{\lambda\mu\nu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (19)$$

and therefore

$$\nabla_{\omega}\, g_{\lambda\alpha\beta} S_{\mu}^{\cdot\,\alpha\beta} = \frac{c^2}{2} \nabla_{\omega}\, g_{\lambda\mu} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (20)$$

Consequently $g_{\lambda\alpha\beta} S_{\mu}^{\cdot\alpha\beta}$ is a tensor, which can differ from $g_{\lambda\mu}$ only by a constant factor. Writing for this factor $-c\varrho$, $\varrho =$ constant, then it follows by substitution in (18)

$$g_{\lambda\mu\nu} = \varrho\, S_{\lambda\mu\nu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Substituting this value in (15), it appears, that two cases are possible, either

$$S_{\lambda\alpha[\mu}\, S^{\alpha}_{\cdot\,\omega\xi]} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

or

$$c = -2\varrho \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

From (7) in relation with (22) we have

$$\overline{\nabla}_{\omega}\, S_{\lambda\mu\nu} = \overset{0}{\nabla}_{\omega}\, S_{\lambda\mu\nu} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

hence the first case gives again the geometries for which $c = +2\varrho$ mentioned above and treated in our previous paper.

For both cases, $c = \pm 2\varrho$, the following relations hold good:

$$\left.\begin{array}{ll} a) & \nabla_{\xi}\, K_{\omega\mu\lambda\nu} = 0 \\ b) & \nabla_{\xi}\, S_{\alpha(\lambda\,|\,\nu\,|}\, S_{\mu)\omega}^{\cdot\,\cdot\,\alpha} = 0 \\ c) & (c-\varrho)\, K_{\omega\mu\lambda\nu} = g_{\omega\mu\lambda\nu} - g_{\omega\mu\nu\lambda} \\ d) & 9\, S_{\omega\mu\lambda\nu}\, S^{\omega\,\cdot\,\cdot\,\nu}_{\cdot\,\xi\eta} = -(c-4\varrho)\, S_{\alpha\mu\lambda}\, S^{\alpha}_{\cdot\,\xi\eta} - 4\, g_{[\mu\lambda]\xi\eta}, \end{array}\right\} \quad . \quad . \quad (25)$$

which are all a consequence of BIANCHI's indentity.

We have therefore as yet to prove, that the case $c = -2\varrho$ leads to the elliptic geometry in S_7. For this it is necessary to use some propositions of the theory of groups.

§ 5. *The group of the* RIEMANN-CHRISTOFFEL *affinor.*

Just as in the previous communication we make use of a system of measure-vectors $\underset{i}{e}^{k}, i, k = 1, \ldots, n$, that is constant by the connexion (—). To two surface-elements in two different points, each defined by two directions, which have in both points the same coordinates x^k, y^k with reference to the system $\underset{i}{e}^{k}$, corresponds the same RIEMANNian curvature, defined analytically by the form

$$R = K_{ijkl}\, x^i\, y^j\, x^k\, y^l \,.$$

V_n admits a translation, whereby every point moves along a linear element, given by a (—)-constant vector [1]). By this translation the change of the vector x^k is given by

$$dx^k = 2\, S_{ij}^{\cdot\cdot\,k}\, x^i\, d\xi^j \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

---

[1]) It may easily be proved that the problem treated above is equivalent to the determination of the V_n, which admit n infinitesimal isogonal translations, not situated in one $(n-1)$-direction.

Since the RIEMANNian curvature of a surface-element does not change by this translation, it follows easily, that the form is invariant by all infinitesimal transformations

$$X_i f = S_{ij}^{\cdot\cdot k} x^j \frac{\partial f}{\partial x^k}; \quad Y_i f = S_{ij}^{\cdot\cdot k} y^j \frac{\partial f}{\partial y^k} \quad . \quad . \quad . \quad . \quad . \quad (27)$$

applied simultaneously on the vectors x^k and y^k.

The transformations (27), whose coefficients because of $c = -2\varrho$ change by the transition from one point of V_n to another, form no group. But they are contained in the group Γ of all rotations which leave the form R invariant. Especially the alternated combinations $(X_i X_j)$ belong to this group[1]):

$$(X_i X_j) = (S_{ik}^{\cdot\cdot\lambda} S_{j\lambda}^{\cdot\cdot l} - S_{jk}^{\cdot\cdot\lambda} S_{i\lambda}^{\cdot\cdot l}) x^k \frac{\partial f}{\partial x^l} \quad . \quad . \quad . \quad . \quad . \quad (28)$$

The differential

$$dx^k = S_{ij}^{\cdot\cdot k} x^i d\xi^j$$

obtained by displacing the vector x^k (0)-parallel, is half the differential (26), whence it follows, that the RIEMANNian curvature is invariant also by this displacement. Hence the V_n belongs to an important class of V_n, viz those in which the curvature is invariant by pseudo-parallel displacements.

Particularly the rotation, corresponding with a surface-element, will belong to the group Γ. This is confirmed by calculation. With a surface-element with the coordinates p^{ij} corresponds the rotation

$$\left.\begin{aligned} p^{ij} K_{ijk}^{\cdot\cdot\cdot l} x^k \frac{\partial f}{\partial x^l} &= p^{ij} ({}^1/_3\, S_{kj}^{\cdot\cdot\lambda}\, S_{i\cdot\lambda}^{\cdot l} - {}^2/_3\, S_{ji}^{\cdot\cdot\lambda}\, S_{k\cdot\lambda}^{\cdot l} + \\ &+ {}^1/_3\, S_{ik}^{\cdot\cdot\lambda}\, S_{j\cdot\lambda}^{\cdot l}) x^k \frac{\partial f}{\partial x^l} = {}^2/_3\, p^{ij} S_{ij}^{\cdot\cdot k} X_k f - {}^1/_3\, p^{ij} (X_i X_j). \end{aligned}\right\} \quad (29)$$

Hence it follows that the group of holonomy of V_n is the group Γ or one of his subgroups[2]).

§ 6. *The group Γ leaves invariant no p-direction.*

It has already been proved in § 2 that there cannot exist a real by Γ invariant p-direction. If there were an invariant imaginary p-direction, the conjugate imaginary p-direction and the orthogonal $(n-p)$-direction would also be invariant. Hence it would follow, that there were an invariant real q-direction, except in the case $n = 2p$ and that the p-direction were totally isotropical. In this latter supposition we choose the system in such a way, that

$$2\, x^1 x^2 + \ldots\ldots + 2\, x^{n-1} x^n$$

---

1) It may be verified easily that X_i and $(X_i X_j)$ does form a group, but we will not use this property.

2) E. CARTAN, Ann. Ec. Norm. 3,42 (25) p. 21. It may be proved, that Γ is the group of holonomy itself.

is the fundamental form, the given p-direction is defined by

$$x^1 = x^3 = \ldots . = x^{n-1} = 0$$

and the conjugate p-direction by

$$x^2 = x^4 = \ldots .. = x^n = 0.$$

Every infinitesimal transformation of Γ contains then only coefficients $a_i^{\cdot j}$ [1], whereby $i+j$ is even and therefore only coefficients a_{ij} whereby $i+j$ is odd. One of the S_{ijk} can therefore only then be equal to zero, if not only $i+j$, but also $j+k$ and $k+i$ are odd.

We find therefore $S_{ijk}=0$.

§ 7. *The group Γ is simple.*

Every linear group which leaves no p-direction invariant is either simple or semi-simple [2].

In the latter case Γ may be obtained from two simple or semi-simple groups γ_1 and γ_2 in q variables $x^1, \ldots, x^q$, resp. s variables $y^1, \ldots, y^s$, which leave invariant no p-direction. The group is then the group of transformations of the $n=qs$ products $z^{(i\alpha)} = x^i\, y^\alpha$, $i=1,\ldots,q$, $\alpha = 1,\ldots,s$ by the transformations of γ_1 and γ_2. Every transformation of Γ is of the form

$$a_i^{\cdot j}\; z^{(i\lambda)} \frac{\partial f}{\partial z^{(j\lambda)}} + b_\alpha^{\cdot\beta}\; z^{(k\alpha)} \frac{\partial f}{\partial z^{(k\beta)}}.$$

Consequently in every vanishing coefficient $A_{(i\alpha)}^{\cdot(j\beta)}$ of an infinitesimal transformation of Γ either $i=j$ or $\alpha=\beta$, moreover, if $i \neq j$ then we have

$$A_{(i\alpha)}^{\cdot(j\alpha)} = A_{(i\lambda)}^{\cdot(j\lambda)}$$

for every value of λ.

The group Γ being a group of rotations, it leaves invariant a non-degenerate quadratic form $\Phi(z^{(i\alpha)})$. Two cases are possible, either Φ vanishes identically or Φ dont vanish. In the first case by giving to the y^α fixed values, we obtain a quadratic form $f(x^i)$, invariant by γ_1, which is determined uniquely but for a constant factor and is not degenerate, since otherwise γ_1 would leave invariant a p-direction. In the same manner a quadratic form $\varphi(y^\alpha)$ may be obtained, so that $\Phi(x^i y^\alpha) = f(x^i)\varphi(y^\alpha)$. We may therefore suppose, that Φ has the form

$$\Phi = \sum_{i,\alpha} (z^{(i\alpha)})^2 \quad \ldots\ldots\ldots\ldots \quad (30)$$

[1] By coefficients of the infinitesimal transformation $e^k X_k f$ we mean the n^2 expressions $e^k{}_{\cdot}S_{k\,i}^{\cdot\,j}$, $i,j=1,\ldots,n$.

[2] To be compared by this and the following §§. E. CARTAN, Les groupes projectifs, qui ne laissent invariante aucune multiplicité plane. Bull. Soc. Math. 41 (13) p. 53—96.

In the second case, setting

$$\Phi\left(z^{(i\alpha)}\right) = g_{(i\alpha)(j\beta)}\; z^{(i\alpha)}\, z^{(j\beta)}$$

it follows

$$g_{(i\alpha)(j\beta)} + g_{(i\beta)(j\alpha)} = 0.$$

The bivector $a_{ij} = g_{(i\alpha)(j\beta)}$, in which expression to α and β fixed values are to be given, is therefore invariant by γ_1 and so is by γ_2 the bivector $b_{\alpha\beta} = g_{(i\alpha)(j\beta)}$, in which i and j have fixed values. These two bivectors are uniquely determined but for constant factors and non-degenerate, for otherwise γ_1 resp. γ_2 would leave invariant a p-direction. The constant factors may be chosen so that for Φ holds good

$$g_{(i\alpha)(j\beta)} = a_{ij}\, b_{\alpha\beta}.$$

Hence it may be supposed that Φ has the form:

$$\left.\begin{aligned} &\Phi = \sum_{i,\alpha} z^{(2i-1,\,2\alpha-1)}\, z^{(2i,\,2\alpha)} - z^{(2i-1,\,2\alpha)}\, z^{(2i,\,2\alpha-1)}, \\ &i = 1, \ldots, \frac{q}{2}; \quad \alpha = 1, \ldots, \frac{s}{2}. \end{aligned}\right\} \quad \ldots \quad (31)$$

Now we will prove that the cases (30) and (31) cannot occur. In the case (30) we have

$$S_{(i\alpha)(j\beta)(k\gamma)} = S_{(i\alpha)\,(j\beta)}^{\cdot\;\;\;\cdot\;\;\;(k\gamma)}.$$

If the coefficient $S_{(i\alpha)(j\beta)(k\gamma)}$ is unequal to zero, then either $j=k$ or $\beta=\gamma$, likewise either $i=j$ or $\alpha=\beta$ and either $i=k$ or $\alpha=\gamma$, therefore either $i=j=k$ or $\alpha=\beta=\gamma$. Now supposing f.i. $i=j=k$ and $\alpha\neq\beta$, then we have for every value of k

$$S_{(i\alpha)\,(i\beta)\,(i\gamma)} = S_{(k\alpha)\,(k\beta)\,(i\gamma)}.$$

For $i\neq k$ the condition mentioned above is not satisfied on the right side of this equation and the trivector S must therefore be equal to zero.

In the case (31) we have

$$S_{(i\alpha)\,(j\beta)\,(k\gamma)} = (-1)^{k+\gamma}\; S_{(i\alpha)\,(j\beta)}^{\cdot\;\;\;\cdot\;\;\;(k'\gamma')}$$

where

$$k' = \begin{cases} k+1, & k \text{ odd} \\ k-1, & k \text{ even} \end{cases}\;;\quad \gamma' = \begin{cases} \gamma+1, & \gamma \text{ odd} \\ \gamma-1, & \gamma \text{ even.} \end{cases}$$

Hence if the coefficient $S_{(i\alpha)(j\beta)(k\gamma)} \neq 0$ we have

$$j=k' \text{ or } \beta=\gamma'; \quad i=k' \text{ or } \alpha=\gamma'; \quad i=j' \text{ or } \alpha=\beta'.$$

Now supposing f.i. $i=j=k'$ with $\alpha=\beta'$, then we have for every value of λ

$$S_{(i\alpha)\,(i\alpha')\,(i'\gamma)} = (-1)^{\alpha+\lambda}\; S_{(i\lambda)\,(i\lambda')\,(i'\gamma)}.$$

For $\lambda\neq\gamma$ we have for every value of k

$$S_{(i\lambda)\,(i\lambda')\,(i'\gamma)} = (-1)^{i+k}\, S_{(i\lambda)\,(k\lambda')\,(k'\gamma)}.$$

Taking $k\neq i$, we must have $\lambda=\gamma'$, i.e. every index λ different from

γ, must be equal to γ'. This however is only possible if $s=2$. If $\lambda=\gamma'$ then we have, supposing $k \neq i$,

$$S_{(i\gamma')\,(k\gamma)\,(k'\gamma)} = - S_{(i\gamma)\,(k\gamma)\,(k'\gamma')}$$

from which it follows $k=i'$. From this we derive that $q=2$ also.

The only possible semi-simple group is therefore the orthogonal group in four variables. The following developments suppose Γ to be simple. They would however be solid also for the orthogonal group in 4 variables. We may however easily verify, that the case $n=4$ cannot occur. Indeed for $n=4$ it follows from (7) that $S_{\omega\mu\lambda\nu}=0$, by which we return to the case $c=+2\varrho$.

§ 8. *The form R derived from the infinitesimal transformations of the group.*

Be

$$Z_I f = a_{Ij}^{\cdot\cdot k}\, x^j \frac{\partial f}{\partial x^k}\,; \quad \begin{matrix} I, J, K, L = 1, \ldots, r, \\ i, j, k, l = 1, \ldots, n \end{matrix}$$

a system of r infinitesimal transformations with real coefficients of Γ. To each of these transformations we adjoin the bilinear form

$$\zeta_I = a_{Ijk}\, x^j y^k = {}^1/_2\, a_{Ijk}\, p^{jk}$$

From the equation

$$(Z_I\, Z_J) = c_{IJ}^{\cdot\cdot K}\; Z_K f \quad \ldots \ldots \ldots \quad (32)$$

it follows that

$$Z_I\, \zeta_J = c_{IJ}^{\cdot\cdot K}\, \zeta_K.$$

Now we introduce the fundamental tensor

$$G_{IJ} = {}^1/_2\, a_{Ijk}\, a_J^{\cdot jk}\ [1] \quad \ldots \ldots \ldots \quad (33)$$

and by using this tensor we obtain from $c_{IJ}^{\cdot\cdot K}$ the covariant components c_{IJK}. From (32) it follows that

$$a_{Ii}^{\cdot\cdot k}\, a_{Jkj} - a_{Ji}^{\cdot\cdot k}\, a_{Ikj} = c_{IJ}^{\cdot\cdot K}\, a_{Kij}\,,$$

from which we obtain by transvection with $a_L^{\cdot ij}$

$$- a_{Ii}^{\cdot\cdot j}\, a_{Jj}^{\cdot\cdot k}\, a_{Kk}^{\cdot\cdot i} = c_{IJK} \quad \ldots \ldots \ldots \quad (34)$$

The c_{IJK} are therefore the components of a trivector.

The form

$$R = K_{ijkl}\, x^i\, y^j\, x^k\, y^l = {}^1/_4\, K_{ijkl}\, p^{ij}\, p^{kl}$$

is a quadratic form in $\zeta_1, \ldots, \zeta_r$. Indeed it follows from (29)

$$\frac{1}{2}\frac{\partial R}{\partial p^{ij}} = \frac{2}{3}\, S_{ij}^{\cdot\cdot k}\, \xi_k - {}^1/_3\, \xi_{ij}, \quad \ldots \ldots \quad (35)$$

where ξ_k and ξ_{ij} are forms which correspond to the infinitesimal trans-

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[1] For this introduction it is evidently necessary that g_{ij} be known.

formations $X_i f$ (27) and $(X_i X_j)$ (28). Now we deduce the general form of a quadratic form $R(\zeta_I)$, invariant by the group Γ. If

$$R(\zeta_I) = A^{IJ} \zeta_I \zeta_J$$

then we have

$$Z_I(R) = A^{JK} \zeta_K c_{IJ}^{\cdot\cdot L} \zeta_L$$

and therefore

$$C_{IJ}^{\cdot\cdot L} \; A^{JK} + C_{IJ}^{\cdot\cdot K} \; A^{JL} = 0.$$

These relations express that the adjoint group of Γ, which is generated by (est engendré par, erzeugt wird durch) the transformations

$$E_I f = c_{IJ}^{\cdot\cdot K} \; e^J \frac{\partial f}{\partial e^K},$$

leaves invariant the form $A_{IJ} e^I e^J$. Consequently, Γ being simple and the adjoint group leaving invariant the form $G_{IJ} e^I e^J$, we have

$$A_{IJ} = h G_{IJ}$$

therefore

$$R(\zeta_I) = h \; G^{IJ} \zeta_I \zeta_J = h \zeta^I \zeta_I \, . \; . \; . \; . \; . \; . \; . \; . \quad (36)$$

From this equation it follows that

$$K_{ij}^{\cdot\cdot ij} = 2hr = -nc; \; h = -\frac{nc}{2r} \; . \; . \; . \; . \; . \; . \quad (37)$$

§ 9. *The order r of Γ is equal to $3n$.*

Taking for the infinitesimal transformations of Γ the n transformations $X_i f$ and $r-n$ other independent ones, then we have for $i, j, k \leqq n$ in consequence of (12)

$$G_{ij} = {}^1/_2 \, S_{i\alpha\beta} \, S_j^{\cdot\,\alpha\beta} = -{}^1/_2 \, c g_{ij} \; . \; . \; . \; . \; . \; . \; . \; . \quad (38)$$

On the other hand it follows from (35) and (36) that

$$\frac{1}{2} \frac{\partial R}{\partial p^{ij}} = h \zeta_I \frac{\partial \zeta^I}{\partial p^{ij}} = \frac{2}{3} S_{ij}^{\cdot\cdot k} \zeta_k - {}^1/_3 \, c_{ij}^{\cdot\cdot I} \; \zeta_I \, .$$

Hence

$$h \frac{\partial \zeta^I}{\partial p^{ij}} = \frac{2}{3} \; \varepsilon_I \, S_{ij}^{\cdot\cdot I} - {}^1/_3 \, c_{ij}^{\cdot\cdot I} \qquad \varepsilon_I = \begin{cases} 1, I \leqq n \\ 0, I > n \end{cases}$$

$$h \frac{\partial \zeta_I}{\partial p^{ij}} = \frac{2}{3} \, G_{Ik} \, S_{ij}^{\cdot\cdot k} - {}^1/_3 \, G_{IJ} \, c_{ij}^{\cdot\cdot J}.$$

By (38) we have

$$h \frac{\partial \zeta_k}{\partial p_{ij}} = h \; S_{kij} = -{}^1/_3 \, c \, S_{ijk} - {}^1/_3 \, c_{ijk},$$

hence

$$c_{ijk} = -(c + 3h) \, S_{ijk} \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (39)$$

Now it follows from (34) that c_{ijk}, $i, j, k \leqq n$, is identical with the quantity g_{ijk} defined by (14) so that we have derived in another way once more the identity (21)

$$g_{ijk} = \varrho\, S_{ijk} \,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

obtaining at the same time as a new result

$$\varrho = -(c + 3h).$$

Now it was proved in § 4 that $c = \pm 2\varrho$. Hence from (37) follows that the only possibilities are

$$c = +2\varrho,\quad h = -\frac{c}{2},\quad r = n$$

giving the V_n of the simple groups, and

$$c = -2\varrho,\quad h = -\frac{c}{6},\quad r = 3n.$$

§ 10. *The group Γ is for $c = -2\varrho$ the orthogonal group in 7 variables.* Now rests only to examine for which types of simple groups, leaving invariant a non-degenerate quadratic form but no linear manifold, r may be equal to $3n$. Previously we remark that the roots of the characteristic equation of an orthogonal group are in pairs equal and opposite. These roots are called by CARTAN the *weights* of the group [1]). It may be remembered that every simple group, which leaves invariant no linear manifold, is entirely determined by her principal weight (poids dominant).

Type A. If the rank of the group is l, then the order is $r = l\,(l+2)$ and every weight is of the form

$$m_1\,\omega_1 + \ldots\ldots + m_{l+1}\,\omega_{l+1}$$

in which the sum of the rational coefficients m_i is zero. The difference between two coefficients, corresponding with the same weight or with two different weights is a whole number. Since with every weight corresponds an equal and opposite weight, the coefficients m_i are either all whole numbers or all half odd numbers. If the m_i are all whole numbers, the group contains all weights $\omega_i - \omega_j$ and 0 and the number of the variables is therefore at least equal to r, which is not possible if $r = 3n$. If the m_i are fractions, then l is odd and all weights of the form

$$\tfrac{1}{2}\left(\omega_1 + \ldots\ldots + \omega_{\frac{l+1}{2}} - \omega_{\frac{l+1}{2}+1} - \ldots\ldots - \omega_{l+1}\right)$$

exist. Their number is therefore

$$\frac{(l+1)!}{\left(\frac{l+1}{2}\right)!\,\left(\frac{l+1}{2}\right)!} > \frac{l\,(l+2)}{3}$$

from which it would follow $n > \frac{r}{3}$.

---

[1]) Loc. cit. § 1, No. 3. We make use of the notations used there.

Type B. Here is $l \geqq 2$, $r = l(2l+1)$ and the weights are of the form

$$m_1 \omega_1 + \ldots\ldots + m_l \omega_l$$

in which the m_i are whole numbers or half whole numbers. In the latter case is $n \geqq 2l > \frac{l(2l+1)}{3} = \frac{r}{3}$. In the first case the existence of a weight, which is not of the form $\pm \omega_i$ has as a consequence that all weights of the form $\pm \omega_i \pm \omega_j$ exist, whence it follows that $n \geqq 2l(l-1) > \frac{l(2l+1)}{2} = \frac{r}{3}$. There remains therefore only the case where the weights have the form $\pm \omega_i$. This case corresponds with the orthogonal group in $2l+1$ variables. From the identity $2l+1 = \frac{l(2l+1)}{3}$ it follows that $l = 3$, so that the orthogonal group in 7 variables appears to be possible.

Type C. Here is $l \geqq 3$, $r = l(2l+1)$ and the weights have the form

$$m_1 \omega_1 + \ldots + m_l \omega_l$$

in which the m_i are whole numbers. The group, for which the weights are $\pm \omega_i$, has $2l$ variables and $2l \neq \frac{l(2l+1)}{3} = \frac{r}{3}$; the other groups have at least r variables.

Type D. Here is $l \geqq 4$, $r = l(2l-1)$ and the weights have the form $m_1 \omega_1 + \ldots + m_l \omega_l$. Is $l \geqq 5$, then the m_i may be half odd numbers, and we have $n \geqq 2^{l-1} > \frac{l(2l-1)}{3} = \frac{r}{3}$. If the m_i are whole numbers, then we have in the first place the system of weights that corresponds with $n = 2l \neq \frac{l(2l-1)}{3} = \frac{r}{3}$ and further other groups, for which $n \geqq r$.

The types E, F, G give groups for which $n > \frac{r}{3}$.

There is consequently but one possibility, Γ is the group of all orthogonal transformations in seven variables. From the expression for R it follows that V_7 is an elliptical S_7.

§ 11. *The absolute parallelism in S_7.*

It is indeed very easy to indicate in S_7 an infinity of parallelisms which satisfy the prescribed conditions. In the projective space of 7 dimensions the absolute be defined by

$$x_0^2 + x_1^2 + \ldots\ldots + x_7^2 = 0.$$

A point be given by 8 coordinates whose squares have the sum 1. The distributive, but not associative numbersystem of GRAVES-CAYLEY with the unities $1, e_1, \ldots, e_7$ is given by the rules of multiplication

$$\left.\begin{array}{l} e_i^2 = -1;\ e_i = e_{i+1}\, e_{i+3} = -e_{i+3}\, e_{i+1} = e_{i+2}\, e_{i+6} = \\ = -e_{i+6}\, e_{i+2} = = e_{i+4}\, e_{i+5} = -e_{i+5}\, e_{i+4} \quad ; \end{array} \quad \begin{array}{l} e_i = e_{i+7} \\ i = 1, \ldots, 7 \end{array}\right\} \quad (40)$$

With the point $x_0, \ldots, x_7$ corresponds the number

$$X = x_0 + \sum_i x_i\, e_i$$

with the module $\sqrt{x_0^2 + x_1^2 + \ldots + x_7^2} = 1$.

$\overrightarrow{XY}$ being an arbitrary segment of a geodesic and X' an arbitrary point, we will call the segment $\overrightarrow{XY}$ and $\overrightarrow{X'Y'}$ aequipollent, if

$$Y'\, X'^{-1} = YX^{-1};\ (X^{-1} = x_0 - \Sigma\, x_i\, e_i) \quad . \quad . \quad . \quad . \quad (41)$$

Equalising the scalar parts of both members of (41) we obtain

$$x_0'\, y_0' + \ldots + x_7'\, y_7' = x_0\, y_0 + \ldots + x_7\, y_7$$

from which follows the equality of length of the segments $\overrightarrow{XY}$ and $\overrightarrow{X'Y'}$. If X and X' are given, the relation between the y_i and the y_i' is linear and such, that with a vector in X corresponds a vector in X' with the same length.

The aequipollence is therefore conformal. The geodesics of S_7 are selfparallel. Indeed, if we put

$$YX^{-1} = Z, \text{ whence } Y = ZX \text{ [1]}$$

and if X' is situated on the geodesic $\overrightarrow{XY}$:

$$X' = \lambda X + \mu Y = \lambda X + \mu ZX$$

then we have

$$Y' = ZX' = \lambda ZX + \mu Z(ZX)$$

so that

$$= \lambda Y + \mu\,(2z_0 - Z^{-1})\, ZX$$
$$= (\lambda + 2\mu z_0)\, Y - \mu X.$$

We get another absolute parallelism by putting

$$X'^{-1}\, Y' = X^{-1}\, Y. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

If generally A is an arbitrary fixed number of the numbersystem, then we have two families of absolute parallelisms, each depending from 7 parameters, by the equations

$$Y'\,(X'^{-1}\, A) = Y\,(X^{-1}\, A) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$
$$(A\, X'^{-1})\, Y' = (AX^{-1})\, Y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

It may be foreseen that there exists an infinity of absolute parallelisms.

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[1]) This result remains valid, although the multiplication is no longer associative.

Indeed, by the infinitesimal translation (26) two (—)-parallel vectors in two different points are transferred into two vectors which are no more (—)-parallel. The metrical properties of S_7 being invariant by translation, the (—)-parallelism is transformed into a (—)-parallelism different from the first, so that there exists a continuous family of (—)-parallelisms.

In S_7 there do not exist other absolute parallelisms than those defined by (43) and (44). Let us consider a determined absolute parallelism and a congruence of geodesics invariant by this absolute parallelism. We get a translation by moving every point on the corresponding geodesic over a distance equal for all points. In an S_n, wherein the equation of the absolute quadric is $x_1{}^2 + \ldots + x_n{}^2 = 0$, every infinitesimal translation may be reduced to the form

$$x_1 \frac{\partial f}{\partial x_2} - x_2 \frac{\partial f}{\partial x_1} + x_3 \frac{\partial f}{\partial x_4} - x_4 \frac{\partial f}{\partial x_3} + \cdots$$

and the trajectories of the translation all intersect the two plane manifolds in the absolute

$$x_1 + ix_2 = x_3 + ix_4 = \ldots\ldots = 0,$$
$$x_1 - ix_2 = x_3 - ix_4 = \ldots\ldots = 0.$$

For $n = 7$ these two manifolds P_3 are situated on the absolute quadric, there exist two different families and two conjugate imaginary manifolds belong to the same family. The P_3 of the first family (P_3^+) are characterised by the property that their equations being written in the form

$$y_0 + iy_1 = y_2 + iy_3 = \ldots\ldots = y_6 + i\, y_7 = 0$$

the determinant of the coefficients of $x_0, \ldots, x_7$ in $y_0, \ldots, y_7$ is positive. By the other family (P_3^-) this determinant is negative.

Reciprocally to every P_3 and his conjugate corresponds a determined congruence of geodesic lines, to which belongs a group of translations with one parameter.

An absolute parallelism may be obtained by choosing ∞^7 manifolds P_3 such that the corresponding congruences are isogonal. It may be proved, that in this manner no other absolute parallelisms are obtained than those which are defined by (43) and (44). The (+)- and (—)-parallelisms are obtained by means of manifolds from (P_3^+) resp. (P_3^-).

The points of S_7, the (+)- and (—)-parallelisms may be considered as elements of S_7. In the same way as we extend in the ordinary projective space the group of projective point-transformations by adjunction of correlations, the group of motions and reflexions in S_7 may be extended by adjunction of the four continuous families of transformations, which transform points in (+)- and (—)-parallelisms. In this manner we have in S_7 a *triality* [1]) by which it is possible to define the distance of two (+)- or two (—)-parallelisms, etc.

[1]) E. Cartan, Bull. Sc. Math. 2, 49 (25) p. 361—371.

§ 11. *General conclusion.*

In a RIEMANNian geometry, in which the linear element is the sum of h linear elements, corresponding with geometries of finite groups and k linear elements, corresponding with the geometry of S_7, there exist, for $k=0$, 2^h absolute parallelisms and, for $k>0$, 2^{h+k} continuous families of ∞^{7k} absolute parallelisms.

This result remains valid, if a euclidian linear element of an arbitrary number of dimensions is added.

P. S. We remark, that an error has been made in the deduction of (10) in the first note, the linear element having really the opposite sign, hence

$$S_{\lambda\mu}^{\cdot\cdot\nu} = + {}^1/_2\, c_{\lambda\mu}^{\cdot\cdot\nu}$$

instead of $-{}^1/_2\, c_{\lambda\mu}^{\cdot\cdot\nu}$. Also on page 807 the linear element $s^j\, dt$ corresponds with the transition from t^k into $t^k - s^i t^j c_{ij}^{\cdot\cdot k}\, dt$, hence in (18) $c_{\lambda\mu}^{\cdot\cdot\nu}$ ought to be substituted by $-c_{\lambda\mu}^{\cdot\cdot\nu}$. The error has had no serious consequences.

Physics. — "*Further measurements on the magnetic disturbance of the supraconductivity with tin and mercury*". (Comm. N⁰. 180 from the Physical Laboratory at Leiden). By W. J. DE HAAS and G. J. SIZOO.

(Communicated at the meeting of May 26, 1926)

§ 1. *Introduction.*

The measurements on the magnetic disturbance of the supraconductivity with tin [1]) and mercury [2]), which have already been published in these Proceedings, had shown, that the magnetic transition lines, measured with decreasing and increasing magnetic fields, do not coincide, as formerly had been assumed, but do form a hysteresis figure.

Besides, it was observed that the transition lines with *mercury*, especially the descending one, show sharp discontinuities, whilst those with *tin* also showed indications of the existence of such discontinuities. The whole of the results led us to assume, that these discontinuities did not correspond to sudden changes in the specific resistance of the metal, but that every jump was due to the disappearance of the resistance of a definite part of the wire. More precisely we supposed, that because of the very slow cooling, in the mercury thread pieces of single crystals would be formed and that a jump would be due to the appearance of supraconductivity in such a single crystal.

We wish to communicate here some new measurements, which partly serve as a completion of those already published and partly may be considered as a confirmation of the given explanation of the discontinuities.

A. Tin.

§ 2. *The hysteresis figure in a longitudinal field.*

Our former measurements on the magnetic disturbance of the supraconductivity with tin, were made with extruded tin wires, which were wound in many turns on a glass tube. Here the direction of the magnetic field was perpendicular to the plane of the turns, and therefore was mainly transverse, that is perpendicular to the direction of the current. It was therefore desirable to measure also the hysteresis figure, for the case of a field, quite parallel to the direction of the current. This was done with the measurement of 17 March 1926.

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[1]) These Proceedings **39**, 221, 1926.

[2]) These Proceedings **39**, 233, 1926.

The resistance used *Sn*-1926-*I*, was made from a tin wire, extruded from "KAHLBAUM" tin, with a diameter of 0.17 mm. The wire was non-inductively wound along a mica strip of 5 cm length. The number of turns, which were separated by silk wire, amounted to 4. Therefore the part of the thread, for which the field was not quite longitudinal, was very small compared to the whole length.

At a temperature of 3°.303 K. the hysteresis figure was measured [1]). The results follow in table I and are represented in fig. 1. The measurement consisted of two series. The points of the two series combine

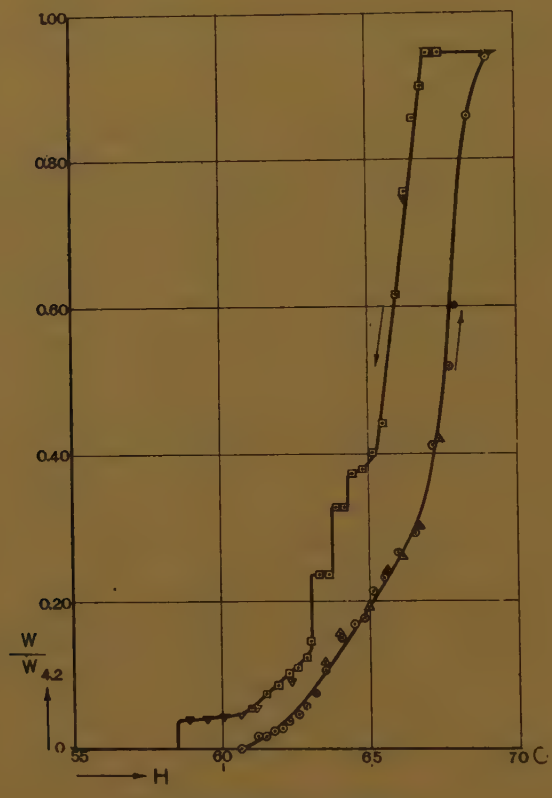

Fig. 1. *Sn*-1926-1 (Table I).

very well to one figure. The figure has a peculiar form, recalling the figures obtained with a transverse field, but still a little different. The descending and the ascending line do not touch each other. Discontinuities are present, especially in the descending line. They are much more distinct than in the transverse field. This is in agreement with the supposition that they are due to the presence of single-crystals. When in an extruded tin wire such single crystals are present, they may easily be destroyed, by winding the wire on a thin glass tube, as was done

[1]) For the method of the measurements, reference may be made to former communications.

TABLE I. *Sn*—1926—*I.* Longitudinal.

$p_{helium} = 264.6$ mm *Hg*. $T = 3^{\circ}.303$ K.

Coil *A*. [1])

$A_{max.} = 0.992.$ $A_{min.} = 0.990,$ $\frac{A_{min.}}{A_{max.}} = 0.991,$

$A_{avg.} = 0.991.$ $H_{avg.}$ (in gauss) = 113.1 i (i in amp.).

Date	Current	$H_{avg.}$ in gauss	$W_{Sn-1926-1}$	$\frac{W}{W_{4.2}}$
17 March 1916	8 mA	63.50	0.00021	0.119
		64.01	0.00028	0.158
		64.98	0 00035	0.193
		65.60	0.00044	0.242
		66.11	0.00048	0.266
		66.73	0.00057	0.315
		67.41	0.00076	0.421
		67.75	0.00132	0.732
		68.60	0.00163	0.904
		69.06	0.00170	0.942
		69.68	0.00170	0.942
		68.09	0.00170	0.942
		67.75	0.00170	0.942
		67.41	0.00170	0.942
		67.13	0.00170	0.942
		66.85	0.00160	0.900
		66.56	0.00155	0.858
		66.28	0.00136	0.753
		66.00	0.00111	0.617
		65.54	0.00079	0.441
		65.15	0.00073	0.403
		64.81	0.00068	0.378
		64.75	0.00068	0.375
		64.13	0.00059	0.329
		64.01	0.00059	0.329
		63.67	0.00043	0.238
		63.45	0.00043	0.238
		63.05	0.00027	0.147
		62.88	0.00023	0.126
		62.60	0.0020	0.112

[1]) These Proceedings **39**, 233, 1926.

TABLE I. *Sn*—1926—*I*. Longitudinal.
(Continued).

Date	Current	$H_{avg.}$ in gauss	$W_{Sn-1926-I}$	$\frac{W}{W_{4.2}}$
17 March 1926	6.7 mA	63.32	0.00019	0.105
		61.92	0.00016	0.088
		61.52	0.00014	0.077
		61.01	0.00010	0.056
		48.55	0.00000	0.000
		60.62	0.000000	0.000
		61.23	0.00003	0.018
		61.47	0.00003	0.018
		61.75	0.00004	0.024
		62.03	0.00005	0.028
		62.32	0.00007	0.038
		62.60	0.00009	0.049
		62.88	0.00011	0.059
		63.16	0.00014	0.077
		63.50	0.00029	0.108
		64.07	0.00027	0.151
		64.52	0.00030	0.168
		64.86	0.00032	0.175
		65.15	0.00039	0.214
		65.54	0.00042	0.231
		66.00	0.00048	0.266
		66.62	0.00053	0.294
		67.24	0.00074	0.413
		67.81	0.00093	0.518
		68.43	0.00155	0.858
		69.11	0.00165	0.918
		69.68	0.00165	0.918
		66.28	0.00135	0.750
		62.32	0.00016	0.091
		61.18	0.00011	0.059
		60.62	0.00008	0.044
		60.05	0.00008	0.044
		59.48	0.00008	0.044
		58.92	0.00008	0.044
		56.65	0.00000	0.000

with the resistances formerly used [1]). When the wire is only stretched along the length of a mica strip, there is much more chance that the crystals remain intact.

§ 3. *Preliminary measurements on the hysteresis figures with single crystals of tin.*

a. A first preliminary measurement with a single crystal wire of tin has already been given in a preceding communication [2]). This wire showed an hysteresis figure, which recalled the figures obtained with extruded tin wires, but differed from those by its greater simplicity. The descending, as well as the ascending line seemed only to consist of two rectilinear pieces.

b. The results obtained with mercury resistances caused us to investigate a tin wire, which first was heated till above its melting point and afterwards very slowly cooled. It might be expected, that with this slow cooling large crystals would be formed, which might give rise to similar

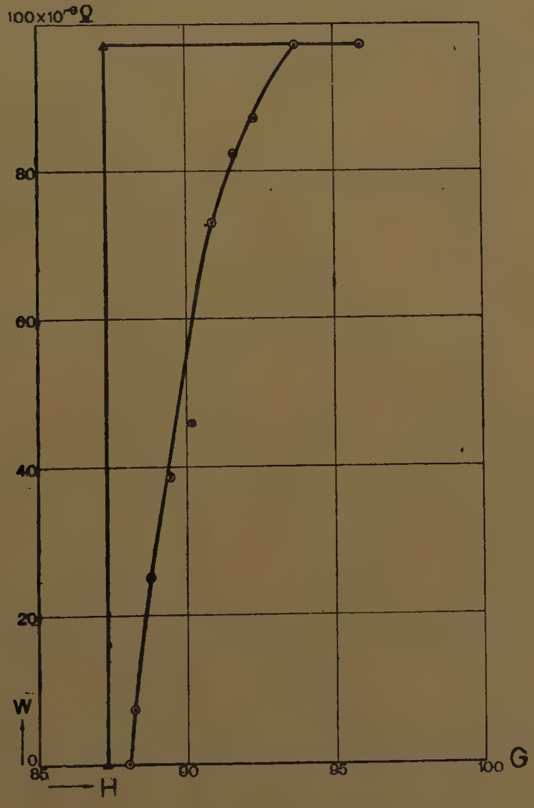

Fig. 2. *Sn*-1926-K_2 (Table II).

1) These Proceedings **39**, 221, 1926.
2) These Proceedings **39**, 253, 1926, fig. 8.

discontinuities as were found with mercury, and, though less distinctly, with tin.

A tin wire, 3.7 cm long, 0.24 mm thick, was fixed between two mica strips, placed in an evacuated glass tube, heated till above its melting

TABLE II. *Sn*—1926—K_2. Longitudinal.

$p_{\text{helium}} = 200$ mm *Hg*. $T = 3°.120$ K.

Coil *W.* [1])

$A_{max.} = 0.783$. $A_{min.} = 0.776$. $\frac{A_{min.}}{A_{max.}} = 0.991$.

$A_{avg.} = 0.780$. $H_{avg.}$ (in gauss) $= 35.87\, i$ (i in amp.).

Date	Current	$H_{avg.}$ in gauss	$W_{Sn-1926-K_2}$	Remarks
17 March 1926	200 mA	104.05 ↓	0.0000978 Ω	
		87.09	0.0000000	jump
		88.02	0.0000000	
		88.23	0.0000073	
		88.73	0.0000251	
		89.45	0.0000387	
		90.17	0.0000459	
		90.88	0.0000730	
		91.60	0.0000823	
		92.31	0.0000872	
		93.67	0.0000971	
		95.89 ↓	0.0000971	
		87.30	0.0000000	jump
	100 mA	95.89 ↓	0.0000971	
		87.0	0.0000000	jump
	40 mA	95.89 ↓	0.0000971	
		87.3	0.0000971	jump

[1]) These Proceedings, 39, 222. 1926.

point and than very slowly cooled. The hysteresis figure was measured on 17 March 1926. See table II and fig. 2.

The result was more surprising than was expected. Namely, whilst the return of the resistance was continuous, the disappearance took place discontinuously, and in fact in one single jump.

We probably may conclude from this, that the whole wire, by the slow cooling, had become a single crystal wire.

The resistance at 0° C. was 0.0798 Ω. The resistance at 4°.2 K. was $98 \times 10^{-6}\ \Omega$. The ratio between these values amounts to 0.00122. With the wire, mentioned under a, this ratio was 0.00027, whilst for extruded tin wires it amounts to about 0.0008. It is evident that this ratio depends largely on the direction of the crystal axis in reference to the length of the wire.

The situation of the jump, namely the value of the field, at which the resistance disappears, was repeatedly determined with different current strengths (200, 100 and 40 mA.). The found differences fall within the accuracy of the measurement. Also with mercury resistances it was always found that the situation of the jumps was independent of the current.

We found no explanation for the difference between the hysteresis figures measured with the two single crystal wires (especially with respect to the ascending line).

We hope to be able to clear up the peculiarities of the magnetic disturbance of the supraconductivity with single crystal wires, by means of new measurements, which are already in progress.

B. Mercury.

§ 4. *The explanation of the discontinuities.*

In a preceding communication [1]) we have already mentioned a measurement, which served to prove the given explanation of the discontinuities by demonstrating that the discontinuous changes of the resistance took place in definite parts of the threads.

Fig. 3.

This measurement was carried out with a resistance of the type, shown in fig. 3. It consists of two capillaries (a and b) connected by a mercury reservoir, provided with platinum wires, so that the resistance of the capillaries might be measured separately (a and b) and together (c). As soon as a sudden change in the resistance was found by decreasing the magnetic field, the resistance of c as well as of a and b was determined. Only with the first jump, a as well as b appeared to have lost a part of their resistance. With all following jumps, the change occurred either in a or in b. The decrease of the resistances in

[1]) These Proceedings **39**, 250, 1926.

a and b with the first jump were not all proportional to the resistances before the jump. There is therefore no question, that the changes in the resistance should be continuously spread over the whole length of both capillaries. By this experiment the local character of the discontinuities might be considered as proved.

TABLE III. *Hg—1926—D.*

Type III. [1]

capillary a: $l=11$ mm. [2] $d=0.10$ mm. [2]

capillary b: $l=11$ mm. $d=0.10$ mm.

$W_{roomtemp.}$ not measured. [3] $W_{4.°2\ K.}$ not measured. [3]

1) $p_{helium} = 624$ mm *Hg*. $T = 4.°001$ K.

2) $p_{helium} = 347$ mm *Hg*. $T = 3.°499$ K.

Current = 40 mA.

Coil *A*. H (in gauss) = 113.1 i (i in amp.).

T	$H_{avg.}$ in gauss	$W_{Hg—1926—D}$ a	$W_{Hg—1926—D}$ b	$W_{Hg—1926—D}$ c	Remarks
3°.001 K.	33.85	0.000031 Ω	0.000051 Ω	0.000085 Ω	
	40.27	0.000064	0.000063	0.000130	
	48.15 ↓	0.000064	0.000064	0.000130	
	30.48 ↓	0.000059	0.000051	0.000110	jump in a and b.
	27.87 ↓	0.000059	0.000000	0.000059	jump in b.
	26.85 ↓	0.000044	0.000000	0.000044	jump in a.
	26.40	0.000000	0.000000	2.000000	jump in a.
3°.499 K.	138.5 ↓	0.000044	0.000042	0.000086	
	123.5 ↓	0.000041	0.000000	0.000041	jump in a and b.
	119.0	0.000000	0 000000	0.000000	jump in a.

To see if the combining of the first jumps was a matter of chance or not, a new measurement was made with *Hg*-1926-*D*, of the same type. See table III and fig. 4. The result was the same. At both temperatures at which was measured the first jump occurs in both capillaries at the same time, the other either in the first or in the second. Also here the

[1] See fig. 3.
[2] l and d represent the *length* and the *diameter* of the mercury thread.
[3] $W_{roomtemp.}$ and $W_{4°.2\ K.}$ represent the resistances of the thread measured at *room-temperature* and at 4°.2 K. respectively.

first changes in the resistances of the capillaries are not proportional to the resistances before the jump.

That there exists a jump, with which both capillaries loose a part of

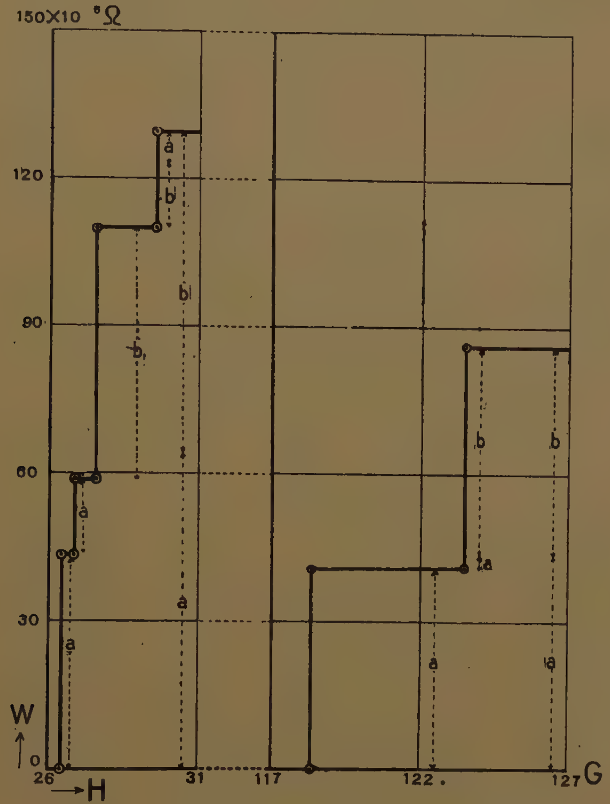

Fig. 4. *Hg*-1926-*D* (Table III).

their resistance, is not to be wondered at. There may very well be formed a single crystal which extends over a part of both capillaries and over the middle reservoir. Why this part prefers to loose its resistance with the first jump, remains however unexplained.

TABLE IV. *Hg*—1926—*A*.

Type II.[1] $l = 10$ mm. $d = 0.10$ mm.

$W_{\text{roomtemp.}} = 0.926\ \Omega$. $W_{4^{\circ}.2\ K.} = 0.000458\ \Omega$.

$p_{\text{helium}} = 9.7 - 10.2$ mm *Hg*. $T = 1^{\circ}.79 - 1^{\circ}.84$ K.

Current $= 28.5$ mA.

Coil *W*.

[1]) These Proceedings 39, 233, 1926.

$A_{max} = 0.783$ $A_{min.} = 0.781.$ $\frac{A_{min.}}{A_{max.}} = 0.99$

$A_{avg.} = 0.782.$ $H_{avg.}$ (in gauss) $= 35.86\, i$ (i in amp.).

$H_{avg.}$ in gauss	$W_{Hg-1926-A}$	$\frac{W}{W_{4.2}}$	Remarks
340.5	0.0000192 Ω	0.0419	
341.9	0.0000258	0.0563	
344.8	0.0000397	0.0867	
345.6	0.0000568	0.1249	
352.7 ↓	0.0000568	0.1249	
339.1	0.0000000	0.0000	jump

TABLE V. *Hg—1926—A*.

See Table IV.

$p_{helium} = 615.0$ mm *Hg* $T = 2°.493$ K.

Current $= 28.5$ mA.

$H_{avg.}$ in gauss	$W_{Hg-1925-A}$	$\frac{W}{W_{4.2}}$	Remarks
278.0	0.000000 Ω	0.0000	
278.3	0.000043	0.095	
293.5	0.000089	0.195	
302.1	0.000090	0.197	
359.2 ↓	0.000090	0.197	
284.5 ↓	0.000026	0.,056	jump
283.7 ↓	0.000007	0.016	"
282.7	0.000000	0.000	"
283.8	0.000002	0.004	
284.8	0.000034	0.074	
287.4	0.000049	0.107	
288.4	0.000061	0.133	
291.0	0.000090	0.197	

§ 5. *The hysteresis figure at different temperatures.*

To find out how the hysteresis figure depends on the temperature, on 11 Febr. 1926, we measured the hysteresis figure of the resistance *Hg*-1926-*A* at four temperatures. The results are contained in the tables IV, V, VI and VII and are represented in fig. 5. As there was

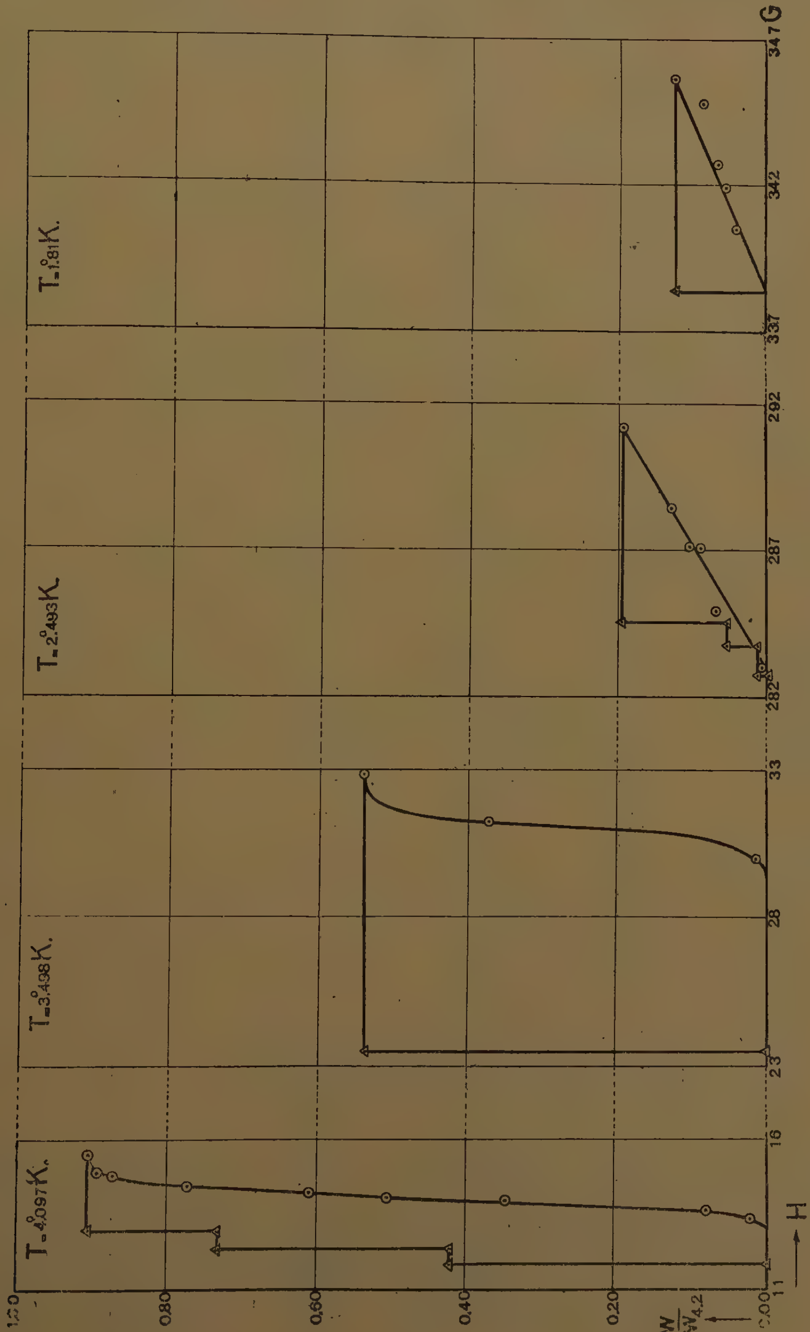

Fig. 5. *Hg*-1926-*A* (Tables IV, V, VI, VII).

TABLE VI. *Hg*—1926—*A*.

See Table IV.

$p_{\text{helium}} = 346.2$ mm *Hg*. $T = 3°.498$ K.

Current = 20 mA.

$H_{avg.}$ in gauss	$W_{Hg-1925-A}$	$\frac{W}{W_{4.2}}$	Remarks
124.3	0.000000 Ω	0.000	
130.0	0.000015	0.000	
131.5	0.000173	0.377	
132.9	0.000249	0.544	
136.1 ↓	0.000248	0.541	
123.5	0.000000	0.000	jump

TABLE VII. *Hg*—1926—*A*.

See Table IV.

$p_{\text{helium}} = 687$ mm *Hg*. $T = 4°.097$ K.

Current = 20 mA.

$H_{avg.}$ in gauss	$W_{Hg-1925-A}$	$\frac{W}{W_{4.2}}$	Remarks
14.37	0.000358 Ω	0.781	
14.91	0.000413	0.902	
15.45	0.000418	0.912	
18.03 ↓	0.000418	0.912	
12.90 ↓	0.000340	0.742	jump
12.36 ↓	0.000196	0.428	,,
11.85	0.000000	0.000	,,
13.34	0.000010	0.023	
13.63	0.000038	0.082	
13.95	0.000162	0.354	
14.04	0.000235	0.513	
14.22	0.000283	0.618	
14.76	0.000403	0.880	
17.96	0.000418	0.912	

little time available we had to measure quickly whence the ascending lines, especially at the lower temperatures, were not so thoroughly determined as was desirable. However, from fig. 5, it is evident that at lower temperatures they are much more inclined to the axis of the field than at the higher temperatures. The distance of the ascending and descending lines measured along the axis of the fields seems to become zero at the high, as well as at the lower temperatures.

The descending line, as is clear from fig. 5, does not always show the same jumps, though this might be expected from the given explanation. However, we have more than once drawn the attention to the fact that two jumps may be combined, by which the form of the descending line may change. The different fields strengths at which the jumps occur, seem to depend on the temperature to a different way, by which also the character of the line may change (see for example fig. 4).

The measurements of the resistance were this time made by comparing the potentials at the ends of the unknown and a known resistance by means of the deflections of a ZERNIKE galvanometer. The proportionality of the deflections to the potentials was secured by the great resistance of the galvanometer compared to the resistances, which were to be measured. Besides it was controlled with the aid of known resistances.

Of the resistance *Hg*-1925-*R* we measured the hysteresis figure at two temperatures (4°.00 and 3°.14 K.). Herewith we found the peculiarity, that a part of the resistance (about 6 %) returned at a much lower value of the field than the rest. Also the disappearance of this part of the resistance look place at an abnormally low value of the field. Besides, in the lower part of the ascending line, there seemed to exist some small discontinuities. For sake of completeness we give the results in tables VIII and IX, but do not wish to attribute a high value to these measurements.

§ 6. *Dependency of the threshold value* $(H\frac{1}{2})$ *on the temperature.*

The threshold value of the magnetic field was defined by TUYN and KAMERLINGH ONNES [1]) as the value of the field, at which the resistance had half returned. This value was called $H\frac{1}{2}$. In accordance to this definition, we now wish to understand by $(H\frac{1}{2})_{asc.}$: the value of the field, at which the resistance, with increasing magnetic field, has reached half of its maximum value at the given temperature, and by $(H\frac{1}{2})_{desc.}$ the value of the field at which the resistance, with decreasing field, has fallen to the same amount. We only give this definition to be able to compare our results with those of previous measurements but do not wish to ascribe any physical meaning to this definition. The peculiar form of hysteresis figure and the discontinuous character of the descending lines make this impossible.

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[1]) Leiden, Comm. No. 174a.

TABLE VIII. *Hg—1925—R.*

Type II. $l=10$ mm. $d=0.05$ mm.

$W_{roomtemp.}=3.96\ \Omega.$ $W_{4^\circ.2\,K.}=0.00242\ \Omega.$

$p_{helium}=625$ mmHg. $T=4^\circ.001$ K.

Current = 4 mA.

Coil *W*.

$A_{max.}=0.783.$ $A_{min.}=0.781.$ $\frac{A_{min.}}{A_{max.}}=0.997.$

$A_{avg.}=0.782,$ $H_{avg.}$ (in gauss) $=35.86\,i$ (i in amp.).

$H_{avg.}$ in gauss	$W_{Hg-1925-R}$	$\frac{W}{W_{4.2}}$	Remarks
7.2	0.00000 Ω	0.000	jump ?
19.32	0.00016	0.066	
19.39	0.00016	0.066	
19.46	0.00016	0.066	
19.61	0.00016	0.066	
19.75	0.00016	0.066	
19.93	0.00016	0.066	jump ?
23.81	0.00024	0.099	
24.06	0.00020	0.083	
26.93	0.00025	0.103	
29.05	0.00030	0.124	
32.68	0.00042	0.174	
33.14	0.00057	0.236	
33.72	0.00113	0.467	
34.04	0.00153	0.632	
35.98	0.00210	0.868	
36.52	0.00208	0.859	
↓ 26.79	0.00168	0.694	jump
↓ 24.32	0.00089	0.368	"
↓ 24.63	0.00027	0.112	"
↓ 14.38	0.00016	0.066	continuous or very small discontinuities ?
13.65	0.00014	0.058	
11.96	0.00010	0.041	
10.77	0.00009	0.036	
8.98	0.00009	0.036	
7.79	0.00005	0.021	
5.74	0.00000	0.000	

TABLE IX. *Hg—1925—R.*

See Table VIII.

$p_{\text{helium}} = 170$ mm *Hg*. $T = 3^\circ.14$ K.

Current = 4 mA.

Coil *W* (see Table VIII).

$H_{avg.}$ in gauss	$W_{Hg-1925-R}$	$\frac{W}{W_{4.2}}$	Remarks
265.7	0.00098 Ω	0.405	
↓ 195.7	0.00057	0.236	jump
↓ 193.9	0.00014	0.056	jump
175.2	0.00006	0.026	continuous or very small discontinuities?
168.1	0.00000	0.000	
190.7	0.00000	0.000	jump?
194.3	0.00012	0.051	
197.5	0.00014	0.062	horizontal?
204.7	0.00013	0.055	
209.9	0.00060	0.246	jump?
211.9	0.00100	0.414	

In fig. 6 the found values for $(H\frac{1}{2})_{\text{asc.}}$ and $(H\frac{1}{2})_{\text{desc.}}$ are plotted as functions of the temperature, together with the older measurements on tin, lead and indium. With those measurements the hysteresis phenomena had not yet been found. As it was to be expected, both lines which represent the connection between $(H\frac{1}{2})_{\text{asc.}}$, and $(H\frac{1}{2})_{\text{desc.}}$ respectively, with the temperature, meet in the same point of the axis of the fields. This point corresponds to the vanishing-point of mercury, namely $4^\circ.17$ K. In comparing these two curves, with the three others, it is to be remembered that our measurements refer to a longitudinal field, whilst the former refer to a transverse field. From some measurements on the threshold values of tin in a longitudinal field we know that the curve $H\frac{1}{2}$, T in a longitudinal field is somewhat steeper than in a transverse field.

According to TUYN and KAMERLINGH ONNES, the curve for tin may be represented by a formula of the form $H\frac{1}{2} = h\,(T_S^2 - T^2)$, where $h = 20.1$ and T_S is the vanishing point of tin ($3^\circ.74$ K.). For mercury a formula of the same form did not appear to hold so well. In fig. 7 the values for $(H\frac{1}{2})_{\text{asc.}}$ are plotted against the corresponding values of T^2.

For comparison the former measurements of tin and indium are also given. It is evident, that the points for mercury do not combine so well to a straight line as those for tin and indium. The straight line, given in the figure, corresponds to the formula: $H_{\frac{1}{2}} = h\,(T_S^2 - T^2)$.

Fig. 6. ($H_{1/2}$, T. Table X).

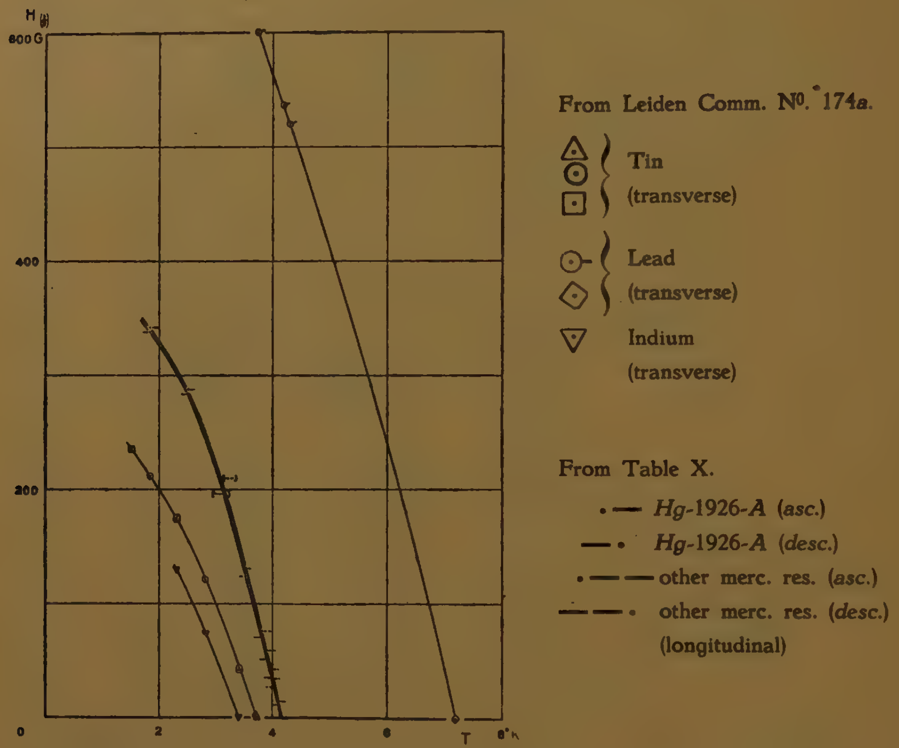
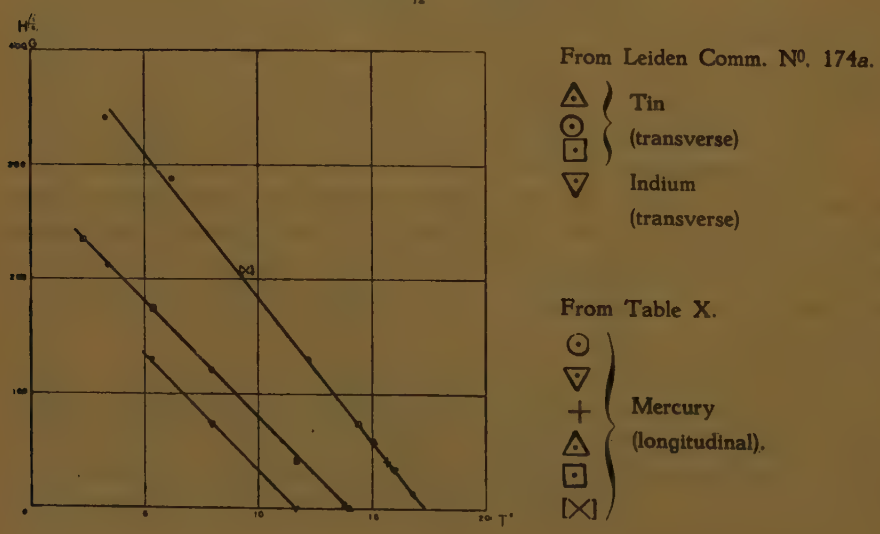


Fig. 7. ($H_{1/2}$, T^2. Table X).



where $h = 25.16$ and $T_S = 4^{\circ}.17$ K. In table X, for all measured resistances, are given the values of $(H\frac{1}{2})_{asc.}$ calculated with this formula and also the measured values of $(H\frac{1}{2})_{asc.}$ and $(H\frac{1}{2})_{desc.}$ The differences between

TABLE X.

Resistance	T	T^2	$(H_{1/2})_{asc.}$ calc.	$(H_{1/2})_{desc.}$ obs.	$(H_{1/2})_{desc.}$ obs.	In fig. 7 indicated by
Hg—1926—*A*	4°.097 K.	16.79	15	14	12	◉
Hg—1925—*C*	4.036	16.29	27	25	23	
Hg—1925—*R*	4.002	16.02	34	34	26	▽
Hg—1926—*D*	4.001	16.01	34	34	28	
Hg—1925—*P*	3.963	15.71	42	42	35	+
Hg—1925—*G*	3.962	15.70	42	42	35	
Hg—1925—*D*	3.924	15.40	50	—	45	△
Hg—1925—*Z*	3.883	15.08	58	59	51	
Hg—1925—*G′*	3.797	14.42	75	75	67	
Hg—1925—*O*	3.797	14.42	75	77.5	64	
Hg—1925—*E*	3.797	14.42	75	77	72	⊡
Hg—1925—*G*	3.796	14.41	75	75.5	67	
Hg—1925—*L*	3.792	14.38	76	77	65	
Hg—1925—*K*	3.792	14.38	76	72	62	
Hg—1925—*A*	3.498	12.24	130	131	124	⊙
[*Hg*—1925—*R*	3.14	9.86	190	210	192] [1]	[X]
Hg—1926—*A*	2.493	6.22	281	288	285	●
Hg—1926—*A*	1.813	3.29	355	342	338	⊙

the measured and the calculated values exceed the accuracy of the measurement by some percents. This is especially evident at the two lowest temperatures at which we measured, with the resistance *Hg*-1926-*A*. For mercury therefore the formula $(H\frac{1}{2})_{asc.} = h(T_S^2 - T^2)$ can only serve as an approximation formula.

[1]) Put between [ ] because of the strange form of the hysteresis-figure (see p. 959).

Chemistry. — *"Equilibria in systems in which phases, separated by a semi-permeable membrane"* XVII. By F. A. H. SCHREINEMAKERS.

(Communicated at the meeting of May 29, 1926).

Ternary systems with vapour-phases.

We now consider an osmotic system:

$$G \mid G_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which a vapour occurs on both sides of the membrane. We represent the composition of the vapour G by:

$$x \; Mol \; X + y \; Mol \; Y + (1-x-y) \; Mol \; W$$

and that of the vapour G_1 by:

$$x_1 \; Mol \; X + y_1 \; Mol \; Y + (1-x_1-y_1) \; Mol \; W.$$

If we assume that the membrane allows the substance W only to pass through, then we find that system (1) is in osmotic equilibrium, when:

$$\zeta - x\frac{\partial \zeta}{\partial x} - y\frac{\partial \zeta}{\partial y} = \zeta_1 - x_1\frac{\partial \zeta_1}{\partial x_1} - y_1\frac{\partial \zeta_1}{\partial y_1} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

As we find, therefore, for the osmotic equilibrium of two vapours the same equations as for the osmotic equilibrium of two liquids, we can say, therefore, that the $O.W.A.$ of a vapour is defined by:

$$\varphi = \zeta - x\frac{\partial \zeta}{\partial x} - y\frac{\partial \zeta}{\partial y} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and consequently we can apply to those osmotic vapour-systems the same considerations as formerly to the osmotic liquid-systems.

Let us take f.i. fig. 1 Comm. II and let us assume that at given temperature and pressure all binary and ternary mixtures are gaseous, then the curves $a\,d$, $e\,h$ etc. can represent isotonic vapour-curves. All vapours of curve $a\,d$ then are mutually isotonic, also those of $e\,h$ etc.; of course vapours of different curves are not mutually isotonic, as the $O.W.A.$ of the vapours of an isotonic curve is larger, the more this curve is situated farther from the point W.

Previously (Communication II) we have deduced that the isotonic liquid curves are in the vicinity of point w (fig. 1. II) straight lines which cut equal parts of the sides WX and WY; on greater distance of point W, however, they are curved, as is drawn in the figure. If we assume, however, as we shall do always in the following, that the vapours follow the law of BOYLE GAY-LUSSAC, then the isotonic vapour-curves

are not only straight lines in the vicinity of the point W, but everywhere. For a vapour, which follows the law of BOYLE GAY-LUSSAC, we have viz.

$$\left.\begin{array}{l}\zeta = x\,C_x + y\,C_y + (1-x-y)\,C_W + \\ \qquad + RT\,[x \log x + y \log y + (1-x-y) \log (1-x-y)]\end{array}\right\} \quad (4)$$

Hence follows for the osmotic equilibrium (1):

$$\varphi = C_W + RT \log (1-x-y) \qquad \varphi_1 = C_W + RT \log (1-x_1-y_1) \quad (4^a)$$

consequently:

$$x + y = x_1 + y_1 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (5)$$

If we give a definite composition to the vapour G_1 then it appears from (5) that the vapour G is represented by the points of a straight line, which goes through the point which represents the point G_1. The isotonic vapour-curves, therefore, are straight lines, parallel to the side XY.

As $1-x-y$ is the concentration of the water in the vapour G and $1-x_1-y_1$ that in the vapour G_1, it follows, therefore:

the vapours G and G_1 have the same $O.W.A.$ if the concentration of the water and, therefore, also the partial vapour-pressure of the water is equal in both vapours.

It now is evident that properties and phenomena, which we have previously deduced, with assumption that the isotonic liquid-curves are curved, are no more valid now. Previously we have seen a.o. that the $O.W.A.$ of the liquid t (fig. 1. II) does not increase immediately with addition of the substance Y; its $O.W.A.$ viz, firstly becomes smaller, till it becomes a minimum in g and only at further addition of Y the $O.W.A.$ increases. Of course a similar phenomenon cannot occur with vapours.

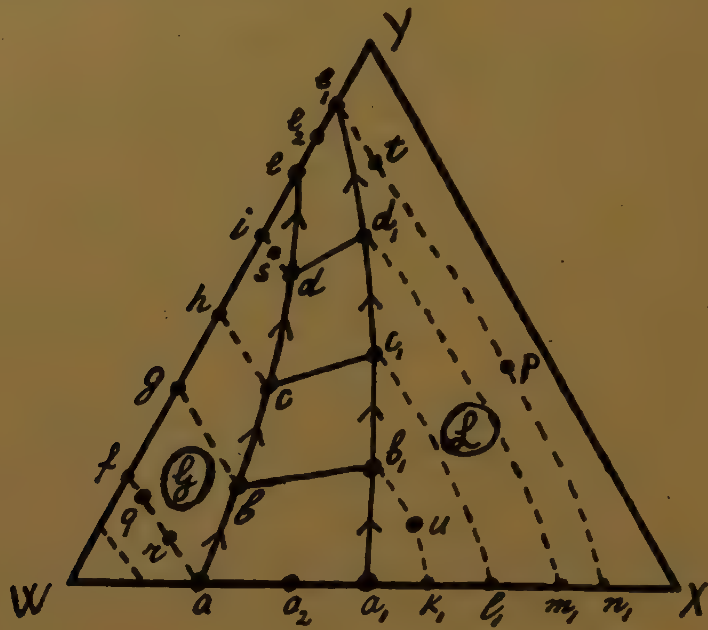
Now we shall assume that with a given temperature and pressure a part of the mixtures is liquid and another part gaseous, so that also may occur equilibria: $G + L$ viz. vapour + liquid. Of the many cases which can occur with this [1]) we shall discuss only some.

If we assume that at the given T and P the substance W is gaseous, but the substances X and Y are liquid, then we may get a diagram as fig. 1, in which Wae represents the vapour-region and $a_1 e_1 YX$ the liquid-region; the first one is indicated by an encircled G, the latter one by an encircled L. Consequently at the given T and P all mixtures of the region Wae are gaseous and those of the region $a_1 e_1 YX$ are liquid. With every vapour of curve $a_1 e_1$ a definite liquid of curve $a_1 e_1$ can be in equilibrium, f.i. vapour a with the liquid a_1; vapour b with

1) For considerations more in detail on ternary systems, in which vapour occurs, compare; F. A. H. SCHREINEMAKERS. Zeitschrift f. Phys. Chem. **36** 257, 413, 710 (1901); **37** 129 (1901); **38** 227 (1901); **43** 671 (1903) Archives Néerlandaises. These Proceedings. Equilibria in ternary systems. Comm. I—XVIII (1912—1915).

the liquid b_1; etc.; some of the vapours and liquids which are in equilibrium with one another are united with one another in the figure by straight lines (conjugation-lines: vapour-liquid).

The field $a\,a_1\,e_1\,e$ situated between the curves $a\,e$ and $a_1\,e_1$ is a heterogeneous field; each point of this field represents viz. a complex of a vapour of curve $a\,e$ and a liquid of curve $a_1\,e_1$; f.i. a point of the line $b\,b_1$ represents an equilibrium $G_b + L_{b1}$; a point of the line $c\,c_1$ an equilibrium $G_c + L_{c_1}$; etc.



The two binary conjugation-lines $a\,a_1$ and $e\,e_1$ of course go through the point W; referring to the position of the ternary conjugation lines we may distinguish two cases, viz.:

a) one (or some) of the conjugation-lines goes through the point W.

b) none of the conjugation-lines goes through the point W.

In fig. 1 the latter case is drawn; with this we have assumed that all conjugation-lines intersect the side $W\,Y$ in points between W and Y. In connection with this assumption we may say, as in the previous communications, that we move away from the point W along the curves $a\,e$ and $a_1\,e_1$ if we proceed along these curves in the direction of the arrows viz. from a towards e and from a_1 towards e_1.

We now take an equilibrium:

$$G + L \quad \ldots \ldots \ldots \ldots \quad (6)$$

of which G and L are represented, therefore, in fig. 1 by two conjugated points, the one of which is situated on curve $a\,e$ and the other on curve $a_1\,e_1$. As G and L are in equilibrium with one another, they are isotonic with respect to all components and, therefore, also with respect to the

diffusing component W. Consequently the vapour a has the same $O.W.A.$ as the liquid a_1; the vapour b the same as the liquid b_1; etc. Consequently we have a.o. the following osmotic equilibria:

$$G_a \mid L_{a_1} \qquad G_b \mid L_{b_1} \qquad G_c \mid L_{c_1} \quad \ldots \ldots \quad (7)$$

in which, however, we can also omit the membrane.

In the previous communication we have deduced:

the $O.W.A.$ of a stable system becomes smaller at taking in of water and greater at losing of water.

If we apply this rule to the equilibria (6) above-mentioned then we find:

the $O.W.A.$ of the vapours of curve $a\,e$ and that of the liquids of curve $a_1 e_1$ increases in the direction of the arrows, viz. from a towards e and from a_1 towards e_1.

In the osmotic system:

$$G_b + L_{b_1} \xrightarrow{\mid} G_c + L_{c_1} \quad \ldots \ldots \quad (8)$$

the system at the right of the membrane has, therefore, a greater $O.W.A.$ than the left system; consequently the water will diffuse in the direction of the arrow, viz. from left to right.

The curves dotted in fig. 1 represent isotonic curves; if we limit ourselves to stable states then we find in the vapour-region $W a\, e$ only isotonic vapour-curves and in the liquid-region $a_1 X Y e_1$ only isotonic liquid-curves. In the heterogeneous region $a\, a_1\, e_1\, e$ all isotonic curves are metastable. The isotonic curves situated in the vapour-region are straight lines parallel to the side XY; the isotonic curves of the liquid-region are more or less curved.

Isotonic curves with a smaller $O.W.A.$ than that of the vapour a are situated within the triangle $W a f$; the isotonic curve with the same $O.W.A.$ as that of the vapour a consists of curve $a f$ and the isolated point a_1. Consequently we can have the osmotic equilibrium:

$$G(af) \mid L_{a_1} \quad \ldots \ldots \quad (9)$$

in which $G(af)$ represents an arbitrary vapour of the curve af. Consequently we may have a.o. the following osmotic equilibria:

$$G_f \mid G_a \;;\; G_q \mid G_r \;;\; G_q \mid L_{a_1} \;;\; G_f \mid L_{a_1} \quad \ldots \quad (10)$$

In the last of those systems a binary vapour, consisting of $W + Y$, is in osmotic equilibrium with a binary liquid consisting of $W + X$.

The isotonic curve with a same $O.W.A.$ as the vapour b consists of two branches viz. of the vapour-branch $g\,b$ and the liquid-branch $b_1 k_1$, which are united with one another by the conjugation-line $b\, b_1$. Consequently all vapours of branch $g\,b$ have the same $O.W.A.$, also all

liquids of branch $b_1 k_1$ and also every vapour of branch $g b$ has the same $O.W.A.$ as every liquid of branch $b_1 k_1$. Consequently we can have the osmotic equilibria:

$$G(g b) \mid L(b_1 k_1) \quad ; \quad G(g b) \mid G_b + L_{b_1} \quad ; \quad L(b_1 k_1) \mid G_b + L_{b_1}$$

in which $G(g b)$ represents a vapour of curve $g b$ and $L(b_1 k_1)$ a liquid of curve $b_1 k_1$. As special case we can have f.i. the osmotic equilibrium

$$G_g \mid L_{k_1}$$

viz. an osmotic equilibrium between a binary vapour with the components $W + Y$ and a binary liquid with the components $W + X$.

If we take isotonic curves with always greater $O.W.A.$, then the vapour-branches approach always more the point e, while the liquid-branches move away always farther from a_1. The isotonic curve with the same $O.W.A.$ as the vapour e consists, besides the isolated point e, still of the liquid-branch $e_1 n_1$. (In reality e is not an isolated point, but the terminating-point of a vapour-branch, which represents on its total length, excepted in the point e itself, only metastable states).

Isotonic curves with greater $O.W.A.$ than that of vapour e (or liquid e_1) consist of one single branch only, which is situated totally within the liquid-region.

We now take the osmotic system:

$$G_q \rightarrow L_p \quad \ldots\ldots\ldots\ldots \quad (11)$$

It appears from the figure that the liquid p has a greater $O.W.A.$ than the vapour q; consequently the water must diffuse in the direction of the arrow viz. from the vapour towards the liquid. Consequently the vapour shifts in the figure along the line Wq away from the liquid and the liquid shifts along the line Wp towards the point W. This diffusion of water continues till vapour and liquid get the same $O.W.A.$ and it depends on the ratio of the quantities of both phases on which isotonic curve this will take place.

We imagine the points p and q to be drawn in the figure in such a way that the line Wp intersects the conjugation-line $b b_1$ and the line Wq the conjugation-line $d d_1$. If the equilibrium is reached on the isotonic curve $g k_1$ then (11) passes into the osmotic equilibrium:

$$G'_q \mid G_b + L_{b_1} \quad \ldots\ldots\ldots\ldots \quad (12)$$

in which G'_q is represented by the point of intersection of Wq and $g b$. The liquid L_p of system (11) is passed, therefore, partly into vapour, on taking up of water. If the equilibrium is formed on the isotonic curve $h l_1$ then (11) passes into the osmotic equilibrium:

$$G'_q \mid L'_p \quad \ldots\ldots\ldots\ldots \quad (13)$$

in which G'_q is represented by the point of intersection of Wq and hc and L'_p by the point of intersection of Wp and $c_1 l_1$.

If the equilibrium is formed on the isotonic curve $i m_1$, then (11) passes into the osmotic equilibrium:

$$G_d + L_{d_1} \mid L'_p \quad \text{. . . . .} \quad (14)$$

in which L'_p is represented by the point of intersection of Wp and $d_1 m_1$.

In the osmotic system:

$$G_r \xrightarrow{|} L_p \quad \text{. . . . .} \quad (15)$$

as is apparent from the figure, the liquid has a greater $O.W.A.$ than the vapour, so that water must diffuse in the direction of the arrow. If we assume that the line Wr intersects the conjugation-line $b\,b_1$ also, then (15) may pass into the osmotic equilibrium:

$$G_b + L_{b_1} \mid G_b + L_{b_1} \quad \text{. . . . .} \quad (16)$$

If the equilibrium is formed on curve $c_1 l_1$ then (15) passes into the osmotic equilibrium:

$$L'_r \mid L'_p \quad \text{. . . . .} \quad (17)$$

consequently into an equilibrium of two liquids which are represented by the points of intersection of the lines Wr and Wp with curve $c_1 l_1$. Therefore, the vapour G_r of system (15) is totally condensed with loss of water.

In the osmotic system:

$$G_r \xrightarrow{|} L_t \quad \text{. . . . .} \quad (18)$$

as is apparent from the figure, the liquid has a greater $O.W.A.$ than the vapour, so that the water diffuses in the direction of the arrow. If we choose the ratio of the quantities of both phases in such a way that the equilibrium is formed on the isotonic curve $h l_1$, then (18) passes into the osmotic equilibrium:

$$L'_r \mid G'_t \quad \text{. . . . .} \quad (19)$$

in which L'_r is represented by the point of intersection of Wr and $c_1 l_1$ and G'_t by the point of intersection of Wt and hc. The result of the diffusion is, therefore, that in (18) the vapour (at the left side of the membrane) passes into a liquid and that the liquid (at the right side of the membrane) passes into a vapour.

We now take an osmotic system of two vapours f.i.

$$G_r \xrightarrow{|} G_s \quad \text{. . . . .} \quad (20)$$

in which, as is apparent from the figure, the vapour s has a greater

$O.W.A.$ than the vapour r. If the equilibrium is formed on the isotonic curve $g\,k_1$ then (20) passes into the osmotic equilibrium:

$$G_b + L_{b_1} \mid G'_s \quad \ldots\ldots\ldots\ldots \quad (21)$$

in which G'_s is represented by the point of intersection of Ws and $g\,b$. The vapour G_r of system (20) is partly condensed, therefore, with loss of water.

The vapour G_r of system (20) can also condense totally; this is the case f.i. if we choose the ratio's of the quantities of the phases in (20) in such a way that the equilibrium is formed on the isotonic curve $h\,l_1$. Then system (20) passes into the osmotic equilibrium:

$$L'_r \mid G'_s \quad \ldots\ldots\ldots\ldots \quad (22)$$

in which L'_r is represented by the point of intersection of Wr and $c_1\,l_1$ and G'_s by the point of intersection of Ws and $h\,c$. The vapour G_r of system (20) is totally converted into liquid, therefore.

At last we still consider an osmotic system of two liquids, f.i.:

$$L_u \xrightarrow{\mid} L_t \quad \ldots\ldots\ldots\ldots \quad (23)$$

in which, as appears from the figure, the water must diffuse from left to right. If we choose the ratio of the quantities of both phases in such a way that the equilibrium is formed on the isotonic curve $h\,l_1$, then (23) is converted into the osmotic system:

$$L'_u \mid G'_t \quad \ldots\ldots\ldots\ldots \quad (24)$$

in which L'_u is represented by the point of intersection of Wu and $c_1\,l_1$ and G'_t by the point of intersection of $W\,t$ and $h\,c$. Consequently the liquid at the right side of the membrane in (23) is totally passed into vapour with taking up water.

In fig. 1 we have assumed that at the given T and P the vapour-region is represented by $W\,a\,e$ and the liquid-region by $a_1\,e_1\,Y\,X$. We now shall suppose, however, the reverse, viz. that $W\,a\,e$ represents a liquid-region and $a_1\,e_1\,Y\,X$ a vapour-region; we have to interchange, therefore, in fig. 1 the encircled letters L and G; also we have to imagine the curves $b_1\,k_1$, $c_1\,l_1$, etc. to be replaced by straight lines, parallel to the side $X\,Y$.

The previous considerations remain all valid now, but of course we have to interchange everywhere vapour and liquid and in the osmotic systems and equilibria (9)—(24) we have to replace the G by L and reversally. If we take f.i. the osmotic system (20) then this now becomes an osmotic system of two liquids viz.:

$$L_r \xrightarrow{\mid} L_s \quad \ldots\ldots\ldots\ldots \quad (25)$$

which, if the equilibrium is formed on the isotonic curve $h\,l_1$, passes into the osmotic equilibrium

$$G'_r \mid L'_s \quad \ldots\ldots\ldots\ldots \quad (26)$$

instead of into (22).

One of the liquids of system (25) viz. L_r now passes with loss of water into the unsaturated vapour G'_r, which is represented in the figure by the point of intersection of Wr and $c_1\,l_1$.

The ternary conjugation-lines are drawn in fig. 1 all in such a way that none of them passes through the point W; if we assume, however, that one of those conjugation-lines f.i. the line $c\,c_1$ goes through the point W, then, if we proceed along the curves $a\,e$ and $a_1\,e_1$, the $O.W.A.$ in c and c_1 is maximum or minimum. We can deduce this in the same way as in communication XIV the corresponding property for the contact of an isotonic curve with the binodal-curve. In previous communications on vapour-tensions in ternary systems [1]) we find many diagrams in which conjugation-lines: vapour-liquid go through the point W.

As in fig. 1 stable states only are indicated, we find the isotonic vapour-curves only in the region $W\,a\,e$ and the isotonic liquid-curves only in the region $a_1\,e_1\,Y\,X$. The first is divided by curve $a\,f$, the second by $e_1\,n_1$ into two parts, which behave themselves differently with respect to osmotic equilibria. A vapour of part $W\,a\,f$ viz. can be isotonic only with vapours, and a liquid of part $e_1\,n_1\,X\,Y$ only with liquids; however, a vapour of part $a\,f\,e$ can be isotonic as well with vapours as with liquids; the same is valid for a liquid of part $a_1\,e_1\,n_1$.

Otherwise it is, however, if we consider metastable states also; then the vapour-region does not terminate in curve $a\,e$ and the liquid-region not in curve $a_1\,e_1$; it depends on the T and P for which the figure is valid, how far both are prolongated. If we choose, as we shall assume further, the T and P in such a way that the components and all their mixtures may be as well liquid as gaseous, then each of both the regions covers the total triangle. Within the region $a\,e\,e_1\,a_1$ then is situated a curve $a_2\,e_2$ of which only the terminating-points a_2 and e_2 are drawn; its interpretation appears from the following. If we leave out of consideration the heterogeneous region $a\,e\,e_1\,a_1$, then all vapours of the region $W\,a_2\,e_2$ and all liquids of the region $a_2\,e_2\,Y\,X$ are stable and consequently all liquids of $W\,a_2 e_2$ and all vapours of $a_2\,e_2\,Y\,X$ are metastable. If we take into consideration, however, also the heterogeneous region, then the vapours of $a\,e\,e_2\,a_2$ and the liquids of $a_1\,e_1\,e_2\,a_2$ are metastable with respect to the equilibria $L+G$.

Of the many cases, which may occur, if we consider metastable phases

[1]) l. c.

also, we shall discuss a few only. We represent the composition of a vapour G by:

$$x \, Mol \, X + y \, Mol \, Y + (1-x-y) \, Mol \, W$$

and that of a liquid L by:

$$x_1 \, Mol \, X + y_1 \, Mol \, Y + (1-x_1-y_1) \, Mol \, W.$$

The thermodynamical potential ζ of the vapour is then defined by (4); in order to represent the thermodynamical potential ζ_1 of the liquid we have to add to (4) still a term which we shall write in the form $RT\mu_1$; further of course we have to replace in (4) x and y by x_1 and y_1. If the vapour is stable with respect to a liquid of the same composition, then is $\zeta < \zeta_1$ and μ_1 is positive, therefore; in the opposite case μ_1 is negative. We now find for the vapour G:

$$\varphi = C_W + RT \log (1-x-y) \quad \ldots \ldots \quad (27)$$

and for the liquid L:

$$\varphi_1 = C_W + RT \log (1-x_1-y_1) + RTm \quad \ldots \quad (28)$$

in which:

$$m = \mu_1 - x_1 \frac{\partial \mu_1}{\partial x_1} - y_1 \frac{\partial \mu_1}{\partial y_1} \quad \ldots \ldots \quad (29)$$

We now take a vapour G_0 of the composition $x_0 \, y_0$ and we consider the osmotic systems:

$$G_0 \mid G \quad . \quad (30^a) \qquad G_0 \mid L \quad \ldots \ldots \quad (30^b)$$

For (30^a) follows from (27) the equation:

$$x_0 + y_0 = x + y \quad \ldots \ldots \quad (31)$$

for (30^b) we find with the aid of (27) and (28):

$$1 - x_0 - y_0 = (1 - x_1 - y_1) \, e^m \quad \ldots \ldots \quad (32)$$

As we can always satisfy (31) also always an osmotic equilibrium (30^a) exists, therefore; that is to say: always a series of vapours exists (totally or partly metastable) which are isotonic with G_0.

However, as we shall see further, it is not always possible to satisfy (32) [viz. by values of x_1 and y_1 which represent points within the triangle]. If we represent the second part of (32) by K then K is a function of x_1 and y_1 which has the value e^{μ_1} in the point W ($x_1 = 0$ and $y_1 = 0$). If we give to x_1 and y_1 such values that the liquid moves away from point W along a straight line going through W, then, as we shall show further, the value of K decreases from e^{μ_1} to zero. (In accordance with fig. 1 we leave here out of consideration dimixtion into 2 of 3 liquids).

We now take the osmotic system (30^b) and we assume, in accordance with fig. 1, that T and P have been chosen in such a way that the watervapour is stable and, therefore, the liquid water is metastable. In point W, therefore, μ_1 is positive and consequently $K = e^{\mu_1} > 1$. As the

first part of (32) is smaller than 1 and as K decreases, when the liquid moves away from W along a line going through point W, consequently on each straight line going through point W is situated a liquid L, which is isotonic with the vapour G_0. Consequently we find:

if at given T and P water-vapour is the stable state of the water, then with every vapour G_0 a series of vapours and a series of liquids can be isotonic.

We now imagine in fig. 1 vapour- and liquid-region to be interchanged then the water is in liquid-state more stable than in vapour-state. In point W, therefore, μ_1 is negative and consequently $K = e^{\mu_1} < 1$. It now depends on the values of x_0 and y_0 which of the two cases

$$1 - x_0 - y_0 < e^{\mu_1} \qquad\qquad 1 - x_0 - y_0 > e^{\mu_1}$$

will occur. In the first case we can satisfy again (32), in the second case, however, not. Hence follows:

if at given T and P the water in stable state occurs as liquid, then with every vapour G_0 a series of vapours can be in equilibrium; however, it depends on the composition of the vapour G_0 whether there is a series of liquids, isotonic with this vapour, or not.

If the liquid L of the osmotic system (30^b) consists of water only, then (32) passes into:

$$1 - x_0 - y_0 = e^{\mu_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

If the liquid water is stable, then is $\mu_1 < 0$ therefore $e^{\mu_1} < 1$; if, however, the liquid water is metastable then is $\mu_1 > 0$ and consequently $e^{\mu_1} > 1$. In the first case, therefore, we can satisfy (33) by positive values of x_0 and y_0, in the second case we cannot do so. Hence follows:

if water is stable in liquid state, then a series of vapours exists, which have the same O.W.A. as this liquid water. (Those vapours, however, are all metastable).

If the vapour G_0 in (30^b) consists of water-vapour only, then (32) passes into:

$$1 = (1 - x_1 - y_1)\, e^{m} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

in which the second part has the value $K = e^{\mu_1}$ for $x_1 = 0$ and $y_1 = 0$. In a similar way as above, it now appears:

if water-vapour is the stable state of the water, then a series of liquids exists, which have the same O.W.A. as this water-vapour. (Those liquids, however, are all metastable).

The property, used above, on the change of K viz. the second part of (32) can be deduced on the following way. It follows viz. from the value of K:

$$\left.\begin{aligned} dK = &-\left[1 + (1 - x_1 - y_1)\left(x\frac{\partial^2\mu}{\partial x^2} + y\frac{\partial^2\mu}{\partial x\,\partial y}\right)_1\right] e^{m}\, dx \\ &-\left[1 + (1 - x_1 - y_1)\left(x\frac{\partial^2\mu}{\partial x\,\partial y} + y\frac{\partial^2\mu}{\partial y^2}\right)_1\right] e^{m}\, dy \end{aligned}\right\} \quad . \quad (35)$$

If we deduce from ζ_1 the values of $r_1\, s_1$ and t_1 then it appears that we may write also for (35):

$$RT\,.\,dK = -(1-x_1-y_1)\left[(rx+sy)_1\,dx_1 + (sx+ty)_1\,dy_1\right]e^m \quad . \quad (36)$$

If we choose dx_1 and dy_1 in such a way that the liquid moves away from W along a line going through point W then is:

$$dx_1 : x_1 = dy_1 : y_1 = d\lambda$$

in which $d\lambda$ is positive; (36) now passes into:

$$RT\,.\,dK = -(1-x_1-y_1)(rx^2+2\,s\,xy+ty^2)_1\,e^m\,.\,d\lambda.$$

If we exclude dimixtion into two or more liquids, then follows that dK is negative, from which follows the above-mentioned property.

We may easily deduce the results obtained in this and previous communications and many other also with the aid of the ζ-surface of the vapours and liquids.

(To be continued).

Histology. — *"Reversible Gelation and Fixation of Tissues."* By Miss M. A. VAN HERWERDEN. (Communicated by Prof. J. BOEKE.)

(Communicated at the meeting of May 29, 1926)

The process of fixation is technically known to the histologist in details, but the physical-chemical event underlying this sudden death of the protoplasm is as yet perfectly unknown.

It seems to me that researches on reversible gelation in the living protoplasm enable us to get an insight into the course of the process of fixation. Experiments in this direction have convinced me that reversible gelation lies on the road to irreversible coagulation, therefore, as far as the tissues are concerned, to *fixation*.

Although there is not yet a constant definition in colloid chemistry of a gel, the most current idea is that the strongly hydrated micellae are united into a threedimensional structure, while inbetween one finds the disperse phase in ultramicroscopical canaliculi. In favour of such a structure (which is easily broken down) is e.g. the possibility — noticed as well in inorganic gels (ironoxide gel) as in living protoplasm — to change suddenly a gel through a simple shaking movement into a less viscid phase (temporary disappearance of mitotic figures a.s.o.).

Having gone beyond the limit of reversibility of such a gel, one has to deal with irreversible coagulation. It is even to be questioned whether there is a sharp limit between gelation and coagulation, whether the change is not a very gradual one. A gel may be regarded as the state of a colloid of which the equilibrium is near coagulation.

Where there is coagulation there is also a tendency to flocculation, as a result of local discharges and dehydration, and a tendency to the formation of filaments (threads of fibrin, tissue fibrils, etc.) Thus one gets a gradual transition of reversible protoplasm gelation to the histological fixation.

In my experiments here following [1]) pointing in this direction, purposely poisons have been chosen which are used in the technique of fixation in histology:

If Paramaecium aurelia is put into 1 % formol, death results. The animals extrude their trichocysts, small rods at the periphery of their body, which, on getting outside, harden into a network of long interwoven stiff threads, a process which often precedes the death of these animals and which reminds one at first of the sudden formation of threads of fibrin. This

[1] These form part of a more extensive research on reversible gelation. (These Proceedings **27**, p. 867 and Archiv. f. exp. Zellforschung, Bd. I, 1925, p. 145).

is doubtlessly a process worth studying from a colloid-chemical point of view, but on which I cannot dwell here.

Thus in 1 % formol an irreversible coagulation takes place, which already may be called fixation although this fixation is better if one increases the percentage, so that death occurs before the trichocysts have been extruded. If, however, Paramaecia of the same culture are put into 0.01—0.1 % formol, a solidification takes place also, but this is no fixation, but a reversible gelation, with which — if only one washes out in time — life is not at stake. The condition returns to normal after washing the animals in ditchwater.

But this is only the case if the formol is soon eliminated, time factor being of considerable influence. It is not without importance that with a solution of gelatine one can do a very similar experiment. If one adds to 9 parts of a 10 % warm solution of gelatine one part of formol, there is no visible change. The gelatine remains transparent even after solidification on cooling the warmed solution has taken place. That the formol, however, has caused a real change in the structure of the gelatine, appears from the fact that on heating the gelatine does not dissolve as before. *The reversible gel has been changed into an irreversible coagulum.*

If, however, heating takes place very soon after adding formol, (e.g. within half an hour) the process is still reversible.

This shows the analogy with what occurs in protoplasm during the process of gelation and coagulation — with this difference, that in the polyphasic protoplasm the relations are of course much more complicated.

The change which takes place in the infusorium may be studied easily — in the dark field as well as with the light field illumination — on adding 0.01—0.1 % formol under the coverglass. The initially optically empty macronucleus becomes quite opaque while the animal moves slowly. The rigidity of the body appears already in its collisions against companions. The ordinary supple, pliable movements are lost.

The foundation of this process on a gelation was even more convincingly demonstrated in another protozoon, Actinophrys sol, which stopped its Brownian molecular movement. I have used this criterion before in another research on leucocytes from the buccal cavity treated with acetic acid [1]). Actinophrys has a round body, surrounded by a halo of fine protoplasmic threads. The type of the different animals changes according to their age and culture medium. In an old culture the animals with coarse opaque granules are preponderant; in younger cultures there are many of which the protoplasm shows a clear Brownian movement. The latter may be stopped by adding a 0.01 % formol solution and restored completely after rinsing out — an example of a reversible change of phase. Simultaneous with the cessation of the Brownian movement Actinophrys detaches from the slide and tends to roll like a ball through the microscopical field.

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a phenomenon which may also be noticed in Amoebae which have been solidificated artificially.

Besides with formol one can also show with another fixative, having a toxic action on the cell, that reversible gelation leads to fixation. With 0.001 % sublimate in water it is namely possible — if quickly rinsed out — to get reversible gelation. With a higher concentration fixation takes place. Just as with formol one can stop with sublimate in Actinophrys the Brownian molecular movement reversibly.

Another fixative commonly used, picric acid, also gives a reversible reaction with a concentration of about 0.01 %, but less easily than formol or sublimate. The time limits between gelation and coagulation approach each other nearer, making it more difficult to observe the phenomenon.

It seems to me that our insight into the nature of fixation is increased by the knowledge that coagulation with a weaker concentration of the fixative goes through a reversible phase. The limits of reversibility are probably fixed by a number of factors among which however — if one experiments with the same substance — concentration is the chief thing. In 0.01 % sublimate the change of gel to coagulum is so rapid that the time limits are too near to intervene experimentally. In 0.001 %, however, the extension of these limits permits to reverse the course of the process and to prolong life, preventing that the change of electric charge of the micellae causes a dehydration and a lasting increase of aggregates, together with a decrease in surface, which is not compatible with the continuation of life.

There are examples in nature in which time, without noticeable outside influence, accomplishes the same change of reversible gel to irreversible coagulum, e.g. the change of silicic acid gels which coagulate in the course of time. Analogies, although of a more complicated nature — owing to the polyphasic nature of the protoplasm — are to be found in living nature in phenomena of senescence.

An important question, the solution of which depends only on the suitability of the material to be tested, is the degree of reversibility in different phases of life. There is every reason to expect that, as life progresses, the reversibility of the started gelation diminishes. Is e.g. the time elapsing between the moment of the still reversible gelation and coagulation different in old and in young cells? — a question which I hope to take in consideration soon. Does this also hold good for processes which take place in the cell normally? It may be expected that also there in the future a similar change has to be tested.

Is not e.g. the increasing cloudiness shown by transparent organisms, which are old or in a state of depression, the manifestation of such an increasing tendency to coagulation, gradually leading to death? Is in such animals — as I described before in fresh water Daphnia — the less transparent diseased organism not to be distinguished at once from the normal, even without this being caused — as shown by measurements — by a general loss of water? One glance at a film with kinematographically

reproduced infusoria — if the eye is accustomed to such matters — is often sufficient to distinguish the animals which have suffered from the ultra-violet light of the arc lamp (perhaps there are some decrepit ones among them) through their cloudy protoplasm and very opaque macronucleus from the normal ones, even if they are still moving about.

That pathological cells show irreversible changes of phase is also known from the aspect of the cloudy swelling. Here also there may be expected that a reversible gelation precedes coagulation. For physiologic-histologic as well as for pathologic research there is an important area, as yet for the greater part unknown, which the biologist may approach in the dawning light on the colloidal properties of protoplasm.

Histology. — *"Day and night period in nuclear divisions."* By C. E. DROOGLEEVER FORTUIN—VAN LEYDEN. (Communicated by Prof. J. BOEKE.)

(Communicated at the meeting of May 29, 1926).

In 1916 I tried to demonstrate that in different tissues of young cats, i.e. in the mesenterium, the epithelium of the cornea, and the epithelium cells of the crypts of Lieberkühn, the number of karyokineses is higher at night than in day-time, consequently there exists a certain periodicity in the number of karyokineses on different hours of the day. Further research showed that also other tissues of the cat, i.e. of the blood producing organs (thymus, lymphgland, spleen, and bone-marrow) showed the same phenomenon with two exceptions. In all cases, the bone-marrow excepted, I found a minimum of nuclear divisions at 10½ a.m. In the bone-marrow too I found at that time a low percentage of dividing nuclei, but the minimum was at 2½ p.m. The maximum of divisions in all cases was at 10½ p.m. or at 2½ a.m. with the exception of the spleen and the thymus which showed a great irregularity at 2½ p.m. These two cases, however, are the only exceptions. Different tissues of tadpoles did not show such a regularity in the nuclear division at different times of the day. Yet I would not be surprised if a more systematic research with short intervals on several days would demonstrate a certain rhythm.

There are several opinions on the periodicity of the nuclear division. I should like to discriminate the periodical and rhythmical nuclear division. The periodical nuclear division then would be the regular alternation of maxima and minima in the number of karyokineses at the same hours of a day. In that case we might speak about a daily period. The rhythmical division, however, should be the occurrence of the division in waves, maxima alternating with minima, independent of day and night.

In different lower plants a periodical nuclear division has been demonstrated. Very well known is the case of Spirogyra in which alga nuclei and cells divide at night. In higher plants less examples are known. ALEXANDER GURWITCH denies the existence of periodicity in the nuclear division. Already in 1910 GURWITCH has ascribed to chance whether in a certain moment a cell would divide or not. The number of causes inducing a cell to divide would be so large that one gets the impression as if the occurrence of the nuclear division is accidental. In tissues always an accidental number of nuclei would be dividing depending upon temperature, light, and several other factors but independent of day and night. In 1924 GURWITCH supposes rays to exist, radiating from the root-

tips and causing the mitotic division. He found those cells to which the rays were directed to be more dividing than others. Also in animal tissues GURWITCH has found those rays, e.g. in the wounded cornea of frogs. Those rays would cause the repair of the wounded tissue. KARSTEN investigated root-tips of Vicia Faba and Zea Mais, cultivated in dark, and vegetation-points of Pisum sativum, Zea Mais, and Pinus Austrica, cultivated in alternating light and dark. He denies an occurrence of a periodicity in the root-tips. In the vegetation-points, however, he demonstrates a maximum of nuclear divisions at night but no distinct minimum as much as I can see from his tables. This period is caused by the regular alternation of day and night. Enlightening during the night and darkening during the day causes two maxima with a difference in time of 12 hours. KARSTEN attributes this phenomenon to the continuation of the inherited old period in some vegetation-points and to direct adaption to the new time of enlightening of other ones. Continuous enlightening eliminates the period and inhibits the growth.

BERINSOHN has investigated the influence of light and dark on the root-tips of Allium Cepa. He found less divisions in daylight than in the dark, if he kept the same onion first in light and afterwards during several hours in the dark. But as he never fixed his material during the night, one cannot draw any conclusion from his tables about the existence of a day and night period. BERINSOHN concludes that in root-tips of Allium Cepa the light inhibits the karyokinesis, in which point he agrees with KARSTEN, when speaking of vegetationpoints of different plants.

A. FISCHER proved photographically that the fibroblasts of an embryonic chicken-heart cultivated in vitro at a temperature of 39° C., are dividing rhythmically. Times of active cell divisions alternate with times of rare divisions or without any division. FISCHER calculates that for his culture the time between two waves of division is about 10 hours. This rhythmic division only occurs in cells of a tissue, not in single cells.

Being of the opinion that the results of my first research indicated the existence of a periodical nuclear division, I decided to continue my research and to investigate, whether still another mammal besides the cat shows a day and night period and which factors have influence upon the process of nuclear divison in tissues. For the first research I chose the mouse, for the second I decided to investigate the influence of light as BERINSOHN has done and besides the influence of changes in the permeability in the cell, the results of which I hope to publish later. As a subject for the latter researches I chose the onion, Allium Cepa, because, as BERINSOHN correctly remarks, experimenting with animals has this drawback that we always have to do with different individuals showing many individual differences whereas the root-tips of one onion belong to one individual.

Before discussing my researches, I wish to point out that the times indicated in this paper differ about half an hour from the real time of the sun, because Peking has the time of the meridian of Shanghai which city is

situated farther to the east than Peking. So in Peking it is in reality always half an hour earlier than the clock indicates.

My method of research was the same as before. Every four hours, i.e. at 11 a.m., 3 p.m., 7 p.m., 11 p.m., 3 a.m., 7 a.m., the different tissues were fixed. Afterwards they were cut, stained, and in one out of 5 sections the total number of nuclei was counted and the number of nuclei in division. The totals of the counted nuclei and of the nuclei in division were taken and the percentage of karyokinesis calculated. If necessary, the probable error was calculated with the formula :

$$\pm \sqrt{\frac{\text{percentage of dividing nuclei} \times \text{percentage of non-dividing nuclei}}{\text{total number of nuclei.}}}$$

On the 1st and 2nd of December 1925, I fixed the small intestine and the cornea of mice, about two weeks old, in the solution of Carnoy. The sections were stained with Heidenhain iron-hematoxylin.

The number of karyokineses in the crypts of Lieberkühn was as follow :

TABLE I.

Time	Percentage of kar.	Number of counted nuclei
3 p.m.	2.008	5030
7 p.m.	2.809	5518
11 p.m.	2.245	4985
3 a.m.	1.508	5105
7 a.m.	3.449	5102
11 a.m.	3.763	5447

Table I shows a series of percentages with the exception of the percentage at 3 p.m. The probable error calculated with the formula given above is

$$\sqrt{\frac{2{,}008 \times 97{,}992}{5030}} = \pm\, 0{,}19.$$

The number 2.008 — 3 times its probable error could still be considered to be the same number. But $2.008 + 3 \times 0.19 = 2.578$, and this number is lower than 2.809. Therefore the percentage of 3 p.m. certainly does not belong to the series.

So we see that the most active nuclear division is in the morning. There is a decrease in the afternoon and a minimum at night. In the cornea there was not a single division. The mice of two weeks were probably too old for this. These numbers in themselves are not very conclusive. Individual differences may be of much influence. Besides it was not certain that the mice were children from one mother as it was in the case of cats, which fact

makes it still more probable that there are individual differences. Yet I believe, if we always get such series with unimportant exceptions, there is an indication of the existence of a periodic nuclear division.

In plants we have the factors more in our own power. Here we can compare the tissues of one individual. Therefore, I put two bulbs of the onion, Allium Cepa, in glasses with water, close to each other in a room not heated, so that they had the same changes of temperatures. One of the onions was kept in full daylight, the other covered by a tin-box so that it was in the dark. When the onions had good strong root-tips, I started my experiment on Dec. 1—2. The weather was cold but the sun was shining. Every 4 hours the root-tips were fixed in corrosive sublimate and sodium chloride and the longitudinal sections later stained with iron-hematoxylin. The counting occurred in the manner described above. I confined myself to the region of the most active growth and the adjacent zone where not so many karyokineses were found but where still an active growth took place. I thought the counting of one half of the root-tip to be sufficient because straight root-tips, which I always choose, have nearly the same number of karyokineses in both halves as GURWITCH has found.

TABLE II. Onion in the dark.

Time	Perc. kar.	Tot. numb. count. nucl.
Dec. 2, 7 a.m.	—	—
Dec. 1, 11 a.m.	2.865	2513
Dec. 1, 3 p.m.	—	—
Dec. 1, 7 p.m.	3.567	1738
Dec. 1, 11 p.m.	6.315	2866
Dec. 2, 3 a.m.	3.779	3096

TABLE III. Onion in daylight.

Perc. kar.	Tot. numb. count. nucl.	Time
3.453	1245	7 a.m. Dec. 2
0.887	789 [1])	11 a.m. Dec. 1
2.486	2292	3 p.m. Dec. 1
—	—	7 p.m. Dec. 1
12.348	1563	11 p.m. Dec. 1
5.069	1634	3 a.m. Dec. 2

Table II shows the observed percentage of karyokineses of the onion in the dark, table III that of the onion in daylight. In both cases we find a striking maximum at 11 p.m. and a minimum at 11 a.m.; at the intermediate times a rise and fall from minimum and the reverse. The differences shown by the onion in daylight are much greater than those of the onion in the dark. I made a second experiment on January 16th—17th, 1926. The onions stood before a window in a room with central heating which caused the differences of temperature not to be great. The weather was bright and the sun shone so that one onion had plenty of light, the other onion again

[1]) This preparation was not very good so that I could only count the karyokineses in some sections. Out of 789 counted nuclei 7 were in division.

was covered with the tin-box. Root-tips were fixed again every 4 hours, now in the solution of FLEMMING, this fluid being more satisfactory than corrosive sublimate and sodium chloride. Staining occurred as before. This time also the different stages of karyokineses were counted. I reckoned all stages before the chromosomes in the aequatorial plate to the prophase, monaster and diaster to the metaphase and the stage of the chromosomes at the poles and two young nuclei to the ana- and telophase.

TABLE IV. Onion in light.

Time	Tot. perc. kar.	Proph.	Metaph.	Ana. teloph.	Tot. numb. count. nucl.	Aver. perc. kar.
Jan. 17, 7 a.m.	11.11	6.28	2.15	2.31	4410	
Jan. 16, 11 a.m.	8.61	4.27	2.32	2.01	3927	6.93
Jan. 16, 3 p.m.	7.57	4.20	1.65	1.72	5933	
Jan. 16, 7 p.m.	6.91	3.33	1.80	1.78	4445	
Jan. 16, 11 p.m.	2.79	1.30	0.76	0.72	5525	
Jan. 17, 3 a.m.	4.58	2.78	0.92	0.88	2837	

In two root-tips of the same onion fixed 5 days at 11 a.m. and 11 p.m., the number of karyokineses was counted in the same manner as before, and I had the surprising result that I found nearly the same numbers as on the corresponding times 5 days earlier as may be seen in table V.

TABLE V. Onion in light.

Time	Tot. perc. of kar.	Proph.	Metaph.	Ana. teloph.
Jan. 21, 11 a.m.	8.85	4.57	1.79	2.48
Jan. 21, 11 p.m.	2.36	1.86	0.11	0.30

Comparing these numbers 8.85 and 2.36 with the corresponding numbers of table IV, 8.61 and 2.79, we see that there is indeed not much difference between the number of karyokineses of the root-tips of the one onion on different days but at the same times.

Two root-tips of the same onion as the one used in table IV fixed at the same time, i.e. 7 a.m., nearly also had the same percentage of karyokineses, i.e. 12.48 and 12.92. The reason why I now found a higher percentage than in table IV at 7 a.m. is that this time I only counted the region just below the top where most of the karyokineses are found and not the adjacent region where the number of karyokineses already decreases. So I counted the nuclei of

the root-tip of table IV at 7 a.m. and found in this manner the percentage to be 12.48 instead of 11.11 in table IV. Therefore we must compare this number with 12.92, the number 11.11 being comparable with the other numbers of table IV. The latter case indicates a synchronic nuclear division in different root-tips of the same plant, whereas table IV and V indicate a regular returning of a daily period.

Table VI shows the percentages of the cells in division of the onion in darkness.

TABLE VI. Onion in darkness.

Time	Tot. perc. kar.	Proph.	Metaph.	Ana. teloph.	Tot. numb. nucl.	Count. aver. perc.
Jan. 17, 7 a.m.	7.78	2.48	1.11	4.19	2170	
Jan. 16, 11 a.m.	8.00	4.97	1.18	1.85	2976	7.4
Jan. 16, 3 p.m.	—	—	—	—	—	
Jan. 16, 7 p.m.	6.70	4.00	1.27	1.44	4728	
Jan. 16, 11 p.m.	7.00	3.37	1.48	2.16	3187	
Jan. 17, 3 a.m.	7.58	5.18	1.07	1.32	3088	

Careful examination of table IV and VI shows for both tables a series, table IV with a maximum at 7 a.m. and a minimum at 11 p.m., the intermediate numbers giving a regular transition between minima and maxima. Table VI gives a maximum at 11 a.m., a minimum at 7 p.m., and between these again a regular transition; here, however, the differences are very small as compared with the differences of table IV of the onion in light. The maximum is 8.00 % and the minimum 6.70 %, whereas 11.11 % and 2.79 % are the maxima and minima of the onion in light.

Table II and III also show these differences. In the onion in light the numbers are varying from 12.34 % to 0.88 %, in the onion in darkness from 6.3 % to 2.86 %.

The times of the maxima and minima do not agree in both cases. In table II and III is a maximum at 11 p.m., in table IV on the contrary a minimum at this time is found and table VI also shows a low percentage then. Table IV shows a maximum at 7 a.m. for the onion in light, table VI a maximum at 11 a.m. for the onion in darkness, at which time tables II and III on the contrary show a minimum.

A third experiment was made on March 18th and 19th at 11 p.m. and 7 a.m. The onions again stood in a room with central heating. On March 19th at 7 a.m. it was almost dark because early in the morning the sky was covered with heavy snow-clouds.

Table VII shows again that the difference of the numbers of karyokineses at different times is larger in the onion in light than in the onion in darkness.

TABLE VII. Onion in light — Onion in darkness.

Time	Tot. perc. kar.	Tot. perc. kar.
March 19, 7 a.m.	11,38	8,35
March 18, 11 p.m.	15,03	7,59

In the latter case the numbers agree with the corresponding numbers of table VI. The numbers of the onion in light are very high. I am sorry not to have fixed root-tips at other times of the day.

Comparing my results with those of BERINSOHN whose tables are copied below (table VIII and IX), a great conformity may be seen.

TABLE VIII. According to BERINSOHN — in light.

Time	Tot. numb. count. nucl.	Tot. numb. mitoses	Spir. loose chromos.	Mon.	Diast.	2 young nuclei
8 a.m.	4000	0,00	0,00	0,00	0,00	0,00
11 a.m.	4345	3,19	1,01	1,01	0,38	0,37
3 p.m.	2290	2,05	1,13	0,61	0,21	0,38

TABLE IX. According to BERINSOHN — in darkness.

Time	Tot. numb. count. nucl.	Tot. numb. mitoses	Spir. loose chromos.	Mon.	Diast.	2 young nuclei
6 a.m.	4702	4,46	3,19	1,4	0,4	0,00
12 m.	4204	4,28	3,49	0,49	0,48	0,09
$6^1/_2$ p.m.	4034	3,06	1,60	0,71	0,24	0,49

Table VIII shows that the difference between maximum and minimum for the onion in light is 3.19 % — 0.00 % = 3.19 %, whereas in table IX it is 4.46 % — 3.06 % = 1.40 % for the onion in darkness.

BERINSOHN concludes from his tables that in the dark the number of karyokineses in the root-tips of Allium Cepa increases, agreeing in this regard with KARSTEN, who says that in the vegetation-points of the stalk of Pisum Sativum and Zea Mais the maximum of embryonic growth is in the darkness of the night. In this regard I agree with both investigators. So

my tables show that the high percentages are at night or early in the morning when in winter there is still twilight, i.e. at 11 p.m. or at 7 a.m. It is possible that the true maximum of division lies earlier in complete darkness and that at 7 a.m. we have already passed this point. KARSTEN, who examined it every two hours, found the highest numbers varying between 9½ p.m. and 6 a.m., whereas at 8 a.m. the number of karyokineses is still high but already lower than the maximum.

In contradistinction to KARSTEN and BERINSOHN, however, I would not conclude that the light does nothing else but inhibit the nuclear division. It is obvious that in general the maxima in the root-tips in daylight are higher and the minima lower than those of the root-tips always kept in dark. Comparing the average percentage of the karyokineses of the root-tips in light with that of the root-tips in dark, we see, that it is 6.93 in table IV against 7.40 in table VI. The average percentage in dark is therefore a little higher than in light. Table VII shows a much higher percentage for the onion in light but this case is not reliable, because we have only regarded the times with the largest numbers of karyokineses for onions in light and not the other times, whereas the percentages of plants in the dark nearly agree with the average at all times. The tables of BERINSOHN contain the same mistake. He only counted the number of karyokineses of the onion in light during day-time and did not regard the hours of the night and early morning, when KARSTEN and I found the maxima.

I conclude that the intermittent daylight and the darkness of the night cause a period in the nuclear division with maxima at night and in the early morning and minima at different times of the day or night. For different individuals the times of maxima and minima are different, for the same individual those times correspond on different days (compare table IV with table V). Probably, ordinarily in tissues, times of active nuclear division alternate with times of spare division and an inclination to a certain rhythm in the division is very likely.

When the light has inhibited the nuclear division for a certain tissue, there will be after the cessation of this stimulus an inclination to an increased division causing more karyokineses in this than there would be, if there had not been an inhibitive influence. We might say that the light inhibits the nuclear division but increases its potency.

Thus, table VI of the onion in the dark also shows a certain rhythm in the nuclear division. I have already drawn the attention to a series with small differences in the number of karyokineses. But calculating the probable error, we see, that the limits of the numbers cover each other. The limits of the highest percentage (8,00) for instance, are

$$8 \pm 3 \times \sqrt{\frac{8 \times 92}{2976}} = 8 \pm 1{,}47,$$

that is 6.53 and 9.47. 6.53 is smaller than 6.70, being the lowest percentage of table VI. So we are not justified in saying that there is a real difference

between maximum and minimum of this table as were in table IV. We no more have a right to conclude to the existence of a daily period.

Yet I will point out a certain rhythm in the different stages of the karyokineses. In table VI we observe at 3 a.m. a large percentage of the cells in the prophase, that is the beginning of the karyokineses (5.18 =), followed by a low percentage in the prophase at 7 a.m. (2.48). At 11 a.m. this percentage has increased to 4.97. I regret, that the next stage which should be low, is lacking. At 7 p.m. the number is rather high (4.00), whereas it decreases a little at 11 p.m. (3.37), and increases at 3 a.m. (5.18). We are justified to say that there is a real difference between all these numbers except between 4.00 and 3.77, the limits of the highest being

$$5.18 \pm 3 \times \sqrt{\frac{5.18 \times 94.82}{3088}} = 5.18 \pm 1.17$$

and of the lowest

$$2.48 \pm 3 \sqrt{\frac{2.48 \times 97.52}{2170}} = 2.48 \pm 0.99.$$

We cannot deny a certain rhythm for the prophase. In the same way there exists a rhythm in the telophase with the difference, that where a maximum is found in the prophase, a minimum in the telophase occurs and the reverse. One gets the impression that nuclei which are at a certain time in the prophase are four hours later in the ana- or telophase. The high percentage of 3 a.m. in the prophase and of 7 a.m. in the telophase for instance, is very remarkable. Tables IV and V show that the percentage of the prophase in root-tips in light follows the same period as that of the total number of karyokineses.

Although it would be very difficult to estimate the duration of one nuclear division, especially because it depends upon temperature and perhaps also upon light, I believe, that table VI shows that here the karyokinesis lasts longer than four hours because we meet a high percentage in the prophase, after four hours again in the telophase. BERINSOHN estimates the time of the division in his root-tips at three to four hours. From my table IV of the root-tips in light, I should also conclude to a time of less than four hours. Differences of temperature and light may be of great influence here.

I agree with BERINSOHN regarding the space of time in which the different stages remain in so far that the time of the prophase must be the longest one. In my own tables and in those of BERINSOHN always the highest numbers are found in the prophase, those of the meta- and ana-telophase being more or less the same, at least in my tables. This is not in accordance with the remark of WILSON (p. 131) "There is reason to conclude that the metaphase is a condition of relative stability in which the mitotic figure often remains for a considerable time."

Conclusions.

I. In the crypts of LIEBERKÜHN of the small intestine of young mice there has been found, with one exception, a daily period for the nuclear division.

II. In the root-tips of Allium Cepa there has been found:

a. Synchronic nuclear division for root-tips of one specimen.

b. In constant darkness a rhythmical division with alternating maxima and minima.

c. Enlightening during day-time changes this rhythm into a daily period with the largest numbers of division in darkness.

d. Although the maxima are higher and the minima are lower in the onion kept in light compared with the one kept in dark, their average percentage of nuclear division is nearly equal.

Finally I wish to thank Dr. DAVIDSON BLACK, Head of the Department of Anatomy of the Peking Union Medical College, for having given me the opportunity to carry out this research.

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Mathematics. — *"On a Function which in any Interval Assumes any Value a Non-Enumerable Number of Times, and on a Function Representing a Rectifiable Curve which in any Interval is Non-Differentiable a Non-Enumerable Number of Times".* By Prof. J. A. BARRAU. (Communicated by Prof. JAN DE VRIES).

(Communicated at the meeting of March 27, 1926).

The communication of J. WOLFF: "On a Function which in any Interval Assumes any Value on a Non-Enumerable Set of Points",[1]) led me to observe that such a function may also be defined in a more elementary way; also M functions y of N variables x so that in the representation of the x space on the y space defined in this way, the image of any N-dimensional region of the former space covers the whole y space a non-enumerable number of times.

The former is done by the aid of the theorem: [2])

"Any number $x\,(0 \leqq x \leqq 1)$ is developed into a binary fraction p (if two developments are possible we choose the one which ends in a repeating zero, not the one with repeating one). If u_n is the arithmetic mean of the first n figures behind the comma in p, for any x the sequence u_n $(n = 1, 2, 3, \ldots$ etc.) has an upper limit y; y is considered as a function of x.

On any sub-interval of the interval $0 \leqq x \leqq 1$ this function assumes all values from 0 to 1 on a set of values of x which has the same power as the continuum" [3]).

For u_n we may choose as well for instance the mean of the first n figures with *odd* order numbers; in this case an arbitrary choice of the figures on the *even* places after a definite order number (hence in a definite interval), has no influence on the value of y (which by a proper choice of the odd places may be made equal to any number from 0 to 1); in this way it is easily seen that the x-values corresponding to the same y, are non-enumerable in the interval in question.

In order to define in a similar way M functions y_i of N variables x_j, we understand by $(u_i)_n$ the mean of the first Nn of those figures in the N developments p_j (for each j the first n) of a point of the x-space of which the order number

$$\varrho \equiv i \,(mod.\, M + 1) \;{}^{4})$$

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1) These Proceedings **29**, p. 127.

2) Published as N⁰. 44, deel XIV, Wiskundige opgaven, Amsterdam, where a proof will be given afterwards.

3) This function has the property that it has any period $\omega = 2^{-N}$ (N a positive integer).

4) We may also assume $\rho \equiv i$ (*mod.* M).

and we put
$$y_i = \overline{\lim_{n=\infty}} (u_i)_n .$$

After a certain order number (hence in a certain sub-hyper cube of the hyper cube $0 \leqq x_j \leqq 1$), we may choose the figures with order numbers $\varrho \equiv i \neq 0$ in such a way that any y_i assumes any desired value from 0 to 1; the choice of the figures with order numbers $\varrho \equiv i = 0$ has no influence on the y_i; hence the power of the set of points x in the sub-hyper cube (and a fortiori in a region containing this), to which there corresponds a given point y, is the same as that of the continuum.

The restriction of x_j and y_i to the values from 0 to 1 is immaterial and is e.g. annulled by repeating the unit-hyper cube of x-space in the directions of the edges and by taking the function

$$z_i = \frac{1-y_i}{y_i} - \frac{y_i}{1-y_i}$$

in stead of any y_i.

A function $y = f(x)$ which represents a rectifiable curve of given length $\lambda > 1$ between the points A and B, (the points 0 and 1 of the X-axis), and which above any sub-interval of AB contains a non-enumerable number of points without tangent, may be defined in the following way:

Between A and B we construct a series of broken lines P_n ($n = 0$, 1, 2, . . . etc.), with lengths

$$l_n = \lambda^{1-2^{-n}}, \quad \text{so that} \quad k_n = \frac{l_n}{l_{n-1}} = \lambda^{2^{-n}}$$

and:

$$\lim_{n=\infty} l_n = \lambda \quad ; \quad \lim_{n=\infty} k_n = 1.$$

P_0 is the line AB.

The odd angular points of P_1 lie on the X-axis in the points 0, $\frac{1}{2}$, $\frac{3}{4}$, . . ., $1-2^{-m}$, . . .

The even angular points lie above the middles of these successive segments, so that P_1 consists of the legs of a series of isosceles triangles. The vertices of these triangles are chosen so that the sum of the legs is successively a_1, $a_1 x_1$, $a_1 x_1^2$, . . ., $a_1 x_1^m$, . . . times their base. Here a_1 is an arbitrary number so that $1 < a_1 < k_1 = l_1$ and x_1 is defined by the condition that P_1 has the length l_1; hence:

$$\frac{a_1}{2} + \frac{a_1 x_1}{4} + \frac{a_1 x_1^2}{8} + \ldots = k_1 = l_1$$

$$2 > x_1 = \frac{2k_1 - a_1}{k_1} = \frac{2\, l_1 - a_1}{l_1} > 1.$$

If φ_m and ψ_m are the angles which (from left to right) resp. the

ascending and the descending leg of the m^{th} triangle make with the X-axis,

$$\lim_{m=\infty} \varphi_m = \frac{\pi}{2} \quad ; \quad \lim_{m=\infty} \psi_m = \frac{\pi}{2}.$$

There is, therefore, a value M_1 of m for which and above which $\varphi > \alpha$, $\psi > \alpha$, if α represents an arbitrary angle in the first quadrant, e.g. $\frac{\pi}{4}$.

On each side of P_1, now from right to left, we choose again the dividing points on $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots$ of this side as odd angular points of a branch of P_2 (with the right extremity of this side as first angular point). The even angular points of this branch of P_2 lie again perpendicularly above the middles of the segments between the odd ones, so that P_2 is formed by the legs of a series of triangles which, in this case, are not isosceles. The vertices of these triangles are chosen so that the proportion of the sum of the legs to the base is successively (from right to left)

$$a_2, \quad a_2 x_2, \quad a_2 x_2^2, \ldots, \quad a_2 x_2^m, \ldots$$

Again

$$1 < a_2 < k_2$$

and

$$2 > x_2 = \frac{2k_2 - a_2}{k_2} > 1,$$

so that the length of P_2 is indeed l_2.

For the angles ψ_m and φ_m which, counted from a sufficiently large m, are surely positive, we have again as above:

$$\lim \psi_m = \frac{\pi}{2} \quad ; \quad \lim \varphi_m = \frac{\pi}{2},$$

hence for $m \geqq M_2$,

$$\psi > \alpha \quad ; \quad \varphi > \alpha.$$

Thus we go on, starting alternately from the left and from the right and always choosing $1 < a_n < k_n$ and $x_n = \dfrac{2\,k_n - a_n}{k_n}$: on each side of P_{n-1} we construct a broken line which is a branch of P_n, and always $l_n = \lambda^{1-2^{-n}}$.

The ordinate of a point x is cut by the lines $P_1, P_2 \ldots, P_n, \ldots$ in points with ordinates $y_1, y_2, \ldots, y_n, \ldots$; these values form a limited sequence which, if x belongs to the binary scale, consists of terms which remain equal from a definite n, but, if x is not a term of the binary scale, increases monotonely. Hence we may put

$$y = \lim y_n \equiv f(x).$$

It is clear that the curve represented by this function, is continuous and rectifiable and has the length λ.

In the points of the binary scale it is neither differentiable on the right nor on the left, for from both sides there approach points as well in a fixed non-perpendicular direction (odd points of a polygonal branch) as in a variable direction, approaching to a perpendicular (even points of a higher branch); the set of these binary points is, however, enumerable.

Any interval β contains also non-binary points which have no differential coefficient. Let us consider a part γ of this interval above which there lies exactly a complete side P_n; such a part may always be found if we choose n sufficiently large.

P_{n+1} has above γ a part consisting of legs of an infinite number of triangles of the same series for which $\varphi > \alpha$; we choose two ascending sides of them (now and in the future always counted from left to right), and call them $\delta_{0,0}$ and $\delta_{0,1}$.

Above each of these sides P_{n+1} has a part for which $\psi > \alpha$; in it we choose two descending sides above each δ: $\varepsilon_{0,00}$ and $\varepsilon_{0,01}$ (above $\delta_{0,0}$), $\varepsilon_{0,10}$ and $\varepsilon_{0,11}$ (above $\delta_{0,1}$). If we go on like this a point x of the interval γ corresponds to the binary development of any number from 0 to 1, around which we may contract an interval so that the chord of the curve above that interval successively ascends more steeply than α and descends more steeply than α; $f(x)$ is, therefore, non-differentiable in that point.

Hence it is evident that the set of these points x on γ (hence a fortiori the set of all the points on β where $f(x)$ is non-differentiable) has the same power as the continuum.

Zoology. — *"On the food of Reef-corals."* By H. BOSCHMA. (Communicated by Prof. C. PH. SLUITER.)

(Communicated at the meeting of June 26, 1926).

Fifty years ago it was generally believed that the food of reef-corals consisted of the plankton organisms floating in the surface water of the sea. The first note on investigations of the food of reef-corals, as far as I am aware, is found in a paper by Sir JOHN MURRAY (1889), based on observations during the Challenger Expedition. He mentions shortly that during this Expedition actual observations on the feeding of corals were made and also investigations on the contents of the gastric cavity. These researches convinced him that the food of the reef-corals consisted of the organisms floating in the sea water.

In the later part of the 19th century KRÄMER (1897) studied the plankton of Samoa and the other regions, as well in tropical as in colder seas. According to him in the Baltic Sea the quantity of plankton is about four times as large as that in the tropics, but he still is convinced that this quantity is sufficient for the feeding of reef-corals.

Now it is well known that reef-corals contain in their entoderm numerous unicellular algae, zooxanthellae. It is generally understood that the association of the two organisms, polyp and algae, is a kind of *symbiosis*, which may be defined as *"a condition of conjoint life existing between different organisms that in varying degree are benefited by the partnership"* (NUTTALL, 1924, p. 213). The nature of this symbiosis has been studied in a number of actinians. BRANDT (1883) made various experiments with *Anthea cereus and Aiptasia diaphana*, actinians which harbour zooxanthellae in their entoderm. Some of these were kept without food in daylight, others without food in darkness. After some months the actinians in daylight were as healthy as at the beginning of the experiment and had but slightly diminished in size. Those which were kept in darkness had after two months lost their zooxanthellae and had died from starvation or were in a decidedly weakened condition. BRANDT concludes from those experiments that the algae in light contributed food in a dissolved form to the actinians. He did not find any zooxanthellae which were being digested (at this time it had recently been discovered where the food of actinians is digested and BRANDT probably did not know this.)

TRENDELENBURG (1909) made accurate gas analyses of the water in which actinians with zooxanthellae in their tissues *(Anemonia sulcata)* lived, as well in darkness as in light. He found that the algae by day furnish oxygen to the polyp and at night utilize oxygen, whilst they derive

their carbon dioxide partly from the animal and partly from the surrounding water. PÜTTER (1911) found that in *Aiptasia* the algae use the nitrogenous waste products of the polyps (ammonia) for the synthesis of their proteins. These researches together with the results obtained by BRANDT show that the association of the actinians with the zooxanthellae has to be regarded as a true symbiosis.

After the study of the contents of the gastric cavity of coral polyps GARDINER (1903) came to the conclusion that the food of reef-corals consists chiefly of their zooxanthellae. In preserved polyps of *Pocillopora* and *"Astraea"* only in 1 or 2 per cent remains of foreign organisms were found. GRAVIER (1908) also was convinced that the food of reef-corals consists for the greater part of the symbiotic algae ; his opinion was based on GARDINER's and his own investigations on the contents of the gastric cavity of the polyps and KRÄMER's statement on the scarcity of the plankton in warmer seas. I came to a similar conclusion (BOSCHMA, 1924) after the study of the contents of the gastric cavity of a great number of East Indian coral polyps. The polyps of smaller size *(Porites, Acropora)* very seldom contained other matter than mucus with zooxanthellae, those of larger size *(Fungia, Favia)* had a mass of mucous matter in their gastric cavity which besides many zooxanthellae consisted of remains of different plankton organisms. I concluded that the food of reef-corals for a considerable part is derived from the zooxanthellae which are digested by the polyps.

These investigations, however, do not prove that the coral polyps derive a part of their nourishment from their zooxanthellae. Coelenterates have no separate system for ingestion and for excretion. These contents of the gastric cavity were for a part certainly food-remnants (the animal remains) but the zooxanthellae might as well have been the products of excretion : the surplus of the quantity needed by the polyps in the tissues. The zooxanthellae are constantly multiplying in the entoderm ; when the cells contain too many algae these are extruded into the gastric cavity. The fact that these algae are found in the gastric cavity does not prove that they are being digested here.

Besides the study of the contents of the gastric cavity feeding experiments have been made on reef-corals. The first who made more or less elaborate investigations of this kind was DUERDEN (1906). He saw that small particles, also plankton organisms, became entangled in the mucus which is secreted on the surface of the polyps of *Fungia* and *Favia*. This mucous layer is afterwards ingested and the food-particles it contains are digested. Much more extensive feeding experiments were made by VAUGHAN (1912) with Floridian and Bahamian reef-coral polyps. Meat of crabs, molluscs or fish, or meat juice was given to the polyps and these substances always brought about the feeding reactions. Also animal plankton (copepods) was eagerly ingested by the polyps. In fact the coral-polyps would feed very readily upon all kinds of animal matter. VAUGHAN also tried vegetable matter : pieces of seaweed and mats of diatoms. This

was invariably refused. Only when the vegetable objects were coated by animal matter it was taken, but soon afterwards the seaweed or the diatoms were removed in an undigested state from the gastric cavity. On the reef also observations were made on corals which had captured animals from the sea-water. VAUGHAN concluded from his experiments: *"The food of reef-corals consists solely of animal matter"* (VAUGHAN, 1912, p. 161). This conclusion in my opinion should be put in a slightly modified form: *"The food of reef-corals as far as it is taken from the outside, in all probability consists solely of animal matter."*

The investigations of METSCHNIKOFF (1880), KRUKENBERG (1880) and WILLEM (1892) showed that the food of actinians is digested in the mesenterial filaments, the free edges of the mesenteries. Next to the border with its numerous nematocysts there is a region the entoderm cells of which ingest small particles in an amoeboid way and digest these particles there. Further particulars on the digestion of actinians are found in papers by MESNIL (1901) and JORDAN (1907). Besides intracellular digestion in the mesenterial filaments there is much evidence for the secretion of a digestive fluid on the larger objects which come into contact with the mesenterial filaments. This fluid causes the disintegration of these objects into small particles, fit to be ingested by the entoderm cells (cf. also BIEDERMANN, 1911).

In 1924 I made some experiments on the feeding reactions and the digestion in the coral *Astrangia danae* in the Marine Biological Laboratory at Woods Hole, Mass. (BOSCHMA, 1925*a*). As in actinians the food is digested in the mesenterial filaments in the region next to the border with its nematocysts. This can be proved by mixing the food with some colouring matter (e.g. India ink, litmus or ammonium carminate). After ingestion of the food the food-vacuoles then contain coloured particles. Most of the colonies of *Astrangia danae* have polyps without any zooxanthellae; the food of these polyps consists exclusively of plankton. On the other hand there are other colonies the polyps of which harbour a great many zooxanthellae in their tissues. In the contents of the gastric cavity of these polyps when freshly collected, besides remains of plankton organisms zooxanthellae are always present. Moreover in the mesenterial filaments of such polyps with zooxanthellae a number of these algae are found in the exact place where after feeding experiments the food is ingested. These zooxanthellae in the mesenterial filaments have lost their natural appearance. Whilst the algae from the entoderm cells of the tentacles and oral disk have a more or less uniform yellow colour, those in the mesenterial filaments contain some brown patches and parts of the cells are completely discoloured. They have undergone here similar changes as occur when zooxanthellae die. It seems to me safe to conclude from this that the disintegration of the zooxanthellae in the mesenterial filaments is due to their being digested here by the polyps.

When polyps containing zooxanthellae are abundantly fed with animal

matter no more zooxanthellae are ingested in the mesenterial filaments. Those already present gradually diminish in size and disappear, evidently by being completely digested, and in the course of a few days during abundant feeding the mesenterial filaments of these polyps become completely devoid of zooxanthellae.

Experiments conducted in the Bermuda Biological Station, (cf. BOSCHMA, 1925*b*) and in the Laboratory of the Carnegie Institution of Washington at Tortugas with various Coelenterates which contain zooxanthellae in their entoderm, viz. reef-corals, actinians, Zoanthids, and Gorgoniids, proved that the mesenterial filaments of these organisms in the natural state always contain a number of zooxanthellae in various stages of disintegration. The symbiotic algae therefore constitute a part of the food of the polyps. As for reef-corals it was possible to make the mesenterial filaments devoid of zooxanthellae by feeding the polyps abundantly with animal food.

To study the role of the zooxanthellae in the feeding of actinians I made some experiments with *Cribrina xanthogrammica* at the Scripps Institution for Oceanography in La Jolla, Calif. The feeding reactions of this species have been studied by GEE (1913). The study of the mesenterial filaments in my experiments gave the following results: Hungry actinians kept in full light ingested many zooxanthellae in the mesenterial filaments and extruded a great number of zooxanthellae in masses of mucus through the mouth. For a month they did not diminish noticeable in size. Though a great number of zooxanthellae were extruded through the mouth the actinians kept the same colour. In diffuse daylight the results were quite similar to those in full sunlight. In general the quantity of zooxanthellae ingested in the mesenterial filaments of hungry polyps in diffuse daylight exceeded that of those in full sunlight. In darkness hungry polyps lost the greater part of their zooxanthellae, thereby assuming a much lighter colour; on the whole these actinians were in an ill state of health, probably owing to the lack of oxygen: as the zooxanthellae could not assimilate in the darkness they could not act as source of oxygen to the tissues of the polyp. The size of these actinians after a month in darkness was about one half of what it originally was. Well fed actinians in all different intensities of light increased in size, though those in darkness were more or less ill, as often bladdery lobes of the stomodaeum were extruded through the mouth, probably due to the lack of oxygen. A result of abundant feeding of the polyp was that as well in light as in darkness no zooxanthellae were ingested in the mesenterial filaments. Great quantities of these algae were removed from the gastric cavity through the mouth. After a month the well fed actinians were often more than twice as large as at the beginning of the experiment.

From these experiments we may conclude the following statement: *The food of reef-corals and of actinians which live in association with zooxanthellae consists for a part of these zooxanthellae and for another part of animal matter. The polyps try to get as much animal matter as*

possible, but in case of starvation they depend chiefly upon the zooxanthellae. The surplus of the rapidly multiplying zooxanthellae in the tissues is removed from the entoderm cells to the gastric cavity, and, as far as needs may be, these algae are digested or removed through the mouth.

The experiments prove that the zooxanthellae are not ingested in the mesenterial filaments because they happen to be in the gastric cavity. The algae are ingested as a source of food, but when sufficient other food is available, they are simply removed through the mouth.

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Chemistry. — *"Optical resolution of chlorobromoacetic acid."* By H. J. BACKER and H. W. MOOK. (Communicated by Prof. F. M. JAEGER.)

(Communicated at the meeting of June 26, 1926).

Chlorobromo- and chloroiodomethanesulphonic acids show a remarkable difference.

POPE and READ [1]) resolved the latter acid into its enantiomorphic components, which were completely stable. They did not, however, succeed in resolving the chlorobromo-compound.

After it had been shown, that in the analogous case of chlorosulphoacetic acid the failure was not due to partial racemy of the alkaloidal salts as had been supposed before [2]), but that it was caused by a strong tendency to racemisation [3]), this same cause was accepted by READ and McMATH [4]) for the chlorobromomethanesulphonic acid. They stated, that under special conditions salts of different rotation were formed, but that in aqueous solutions only inactive mixtures could be obtained.

This striking difference between the chlorobromo- and chloroiodo-compounds raises the question, whether perhaps the close chemical analogy of chlorine and bromine might prevent resolution or promote racemisation.

In the literature we only found two examples of resolvable chlorobromo-carboxylic acids, the asymmetry of which is due to the carbon atom bearing chlorine and bromine, namely fluorochlorobromoacetic acid $CFClBr.CO_2H$ [5]) and methylchlorocyclohexyl-chlorobromoacetic acid $C_7H_{12}Cl.CClBr.CO_2H$ [6]).

In both acids the asymmetric carbon atom has no hydrogen atom and yet in the first case the tendency to racemisation is so strong, that the active acids could not be separated from the alkaloid salts of different rotation.

To test the influence of chlorine and bromine attached to an asymmetric carbon atom, we have now examined one of the simplest examples namely chlorobromoacetic acid, $ClBrCH.CO_2H$.

This compound, which is closely analogous in structure with chlorobro-

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1) Journ. chem. Soc. **105**, 811 (1914).

2) POPE and READ, Journ. chem. Soc. **93**, 796 (1908).

3) BACKER and BURGERS, These Proc. **28**, 64 (1925).

4) Journ. chem. Soc. **127**, 1572 (1925).

5) SWARTS, Bull. Ac. r. Belg. [3], **31**, 28 (1896); Mém. cour. Ac. Belg. **54** (1896).

6) PERKIN and POPE, Journ. chem. Soc. **99**, 1527 (1911).

momethanesulphonic acid $ClBrCH.SO_3H$, may be prepared from commercial trichloroethylene [1]).

To effect its resolution the method of „cold crystallisation" [2]) was applied to various alkaloidal salts.

With brucine a laevorotatory acid was obtained, whilst quinine gave a dextrorotatory compound.

The latter alkaloid gave the best results.

Since neutralisation of the carboxyl group in the sulphocarboxylic acids causes a considerable change in molecular rotation, it was important to determine the influence of neutralisation in this case.

It was found, that the salts show a rotation of about the same value as that of the acid, but in the opposite direction.

The highest molecular rotation for the D-line, so far observed, is $+8°$ for the free acid and $—8°$ for the ammonium salt.

Further experiments on the dispersion are in progress, which may perhaps give a greater rotation.

In the first place we were interested in the tendency to racemisation; it proved to be not so great as might have been expected.

A solution of the ammonium salt did not racemise in 24 hours, even in the presence of a molecule of sodium hydroxide.

After heating for half an hour on a water bath the alkaline solution was half racemised.

For a solution of the ammonium salt, containing 0.089 G.mol. per litre, kept at ordinary temperature from October 1925 until June 1926, the rotation had diminished to half the original value.

Thus a compound with chlorine and bromine, attached to an asymmetric carbon atom, has now been resolved into its enantiomorphs ; further in the case examined racemisation of the active components was slow.

Org. Chem. Laboratory of the State University.

Groningen, June 1926.

1) Method of CROMPTON, Compare W. H. VAN MELS, Dissertation Groningen, 26 June 1926, p. 48.

2) These Proc. 28, 64 (1925).

Physiology. — *"On the permeability of the gut in seacucumbers."* By H. A. P. C. OOMEN. (From the Physiological Department of the Zoological Station at Naples.) (Communicated by Prof. R. MAGNUS.)

(Communicated at the meeting of October 31, 1925).

In 1901 COHNHEIM who previously had demonstrated *active* resorption to occur in the surviving gut of mammals (1) published a paper on "Resorption, Verdauung und Stoffwechsel der Echinodermen" (2). Among other things he states that the Holothurian gut, though, with regard to solved substances, it behaves in general like a simple diffusion membrane, is capable to resorb isotonic solutions from its lumen. His methods of experimentation and results obtained thereby have been severely criticized by ENRIQUES (3). Nevertheless they were accepted and in the literature concerning cellpermeability his observations are cited as remarkable examples (e.q. HÖBER 1924). The results of ENRIQUES' did not seem to agree with the main function of an intestinal tract. He found the intestinal wall to be absolutely semi-permeable in Holothurians: it allows water to pass, but no solved substances. He denied the existence of active resorption from the isolated surviving gut. These two paradoxical properties render the answer to the question as to how nutritive substances to be resorbed, pass through the intestinal wall, rather difficult.

Neither do the results obtained by ENRIQUES seem to agree with the data furnished bij other Evertebrates. In general we may distinguish two different ways in which substances present in the intestinal tract may reach the blood and tissues. The first of these is the above mentioned true resorption, a polarised process by which solved and solving substances may disappear from the intestinal lumen, without or against a difference in osmotic pressure. In other cases diffusion occurs; the intestinal contents pass through the wall as they would through a membrane of parchment paper, according to the laws of diffusion.

The difference between the two is very evident in experiments by H. J. JORDAN and H. BEGEMANN (4) on frogs and snails, who in a living frog filled the gut with solutions, hypo- or isotonic with regard to the blood, and then ligated them at both ends. Two or three days afterwards the contents had disappeared completely. The gut of the snail, however, behaves quite differently. During the same lapse of time the volume of the intestinal contents remains practically the same, but part of the solved glucose has disappeared, evidently by diffusion.

In experiments on the physiology of digestion in sea-cucumbers, made

at Naples, I tried to decide whether COHNHEIM or ENRIQUES were right and furthermore to find out in which way solved substances pass the intestinal wall in these animals.

It proved to be practically impossible to fill and ligate the gut in the living animals; the animals autotomise the whole intestinal tract and its appendages on nearly every strong stimulus. I therefore was obliged to work on isolated surviving guts [1]). The experiments were carried out as follows: the animals *(Holothuria tubulosa and Stellati)* were carefully dissected and their guts procured. With fine scissors I then removed the wonder-nets and other vessels from the intestinal wall. I then removed the contents of stomach and gut, rinsed them if necessary, ligated one end with a strong thread and filled them with the test-solution from a buret. Then the other end was ligated and the preparation was immersed very carefully in a small container, the contents of which were ventilated and early replaced by running water if the experiment allowed to act thus. At the end of the experiment I lifted the organ by means of the two threads, dried it with filter-paper (not too carefully, in order to be able to notice weak spots) and returned the contents to a buret, set up previously. Control experiments have shown that one must not continue the experiments for longer than 18 hours, if one wishes to be sure to experiment on a surviving and not on a dying gut. After this lapse of time and when the ventilation had not well been taken care of, the results soon become very irregular.

In the first place I had to investigate whether in our animals the surviving gut is capable of resorption. I therefore took a number of stomachs and guts, filled them in the above mentioned way with seawater and suspended them in running sea-water. (The fluid of Holothurians is nearly always isotonic with the medium in which they live.) The following table shows the status at the end of an experiment of six hours.

Object	Length in cm.	Volume in cc		Total Cl expressed in cc 0.2 N.$AgNO_3$	
		before	after	before	after
intestine	25	1.7	1.8	5.5	5.6
stomach	25	7.1	7.1	22.9	23.3
intestine	35	5.8	5.6	18.7	18.3
stomach	30	3.0	3.4	9.7	10.3
stomach	40	9.1	9.2	29.3	30.0

The initial volume was determined by the readings of the filling buret, the final volume is the quantity found at the end of the experiment. The

1) For convenience' sake, although in the intestinal tract of Holothurians one may distinguish a distinct stomach and (end-) gut, I use the word intestinal tract or gut, where there is no need of discrimination.

percentage of chlorids was determined by MOHR's titration with $AgNO_3$. 1 cc 0.2 N. $AgNO_3$ = 7.092 mg Cl. Limit of error with regard to volume ± 0.15 cc, with regard to the Cl percentage ± 0.3 cc, 0.2 N. . $AgNO_3$.

Neither the volume, nor the percentage of chlorids in this series of experiments show any variation outside the limit of error. I therefore conclude that under proper precautions there is no permeability in either direction and no resorption. If, however, one neglects the limit of duration on the oxygen provision or if one fills the organs to the limit of their capacity one finds a diminution of volume and total chlorides. With my experimental set-up, one hardly obtains positive results with such abnormal organs. They become soft and fragile, the epithelium dies and disappears and they nearly always break when lifted from the water and put on the filter paper.

The next thing to find out was whether and how solved substances are influenced by the intestinal wall, separating two different solutions. For this purpose I chose in the first place different concentrations of NaCl.

The following solutions were used: 1⁰. sea-water; 2⁰. four parts of sea-water, one part of distilled water; 3⁰. sea-water and 0.8 % NoCl.

1. Seawater outside, diluted seawater inside. Duration of the experiments 6 hours.

Object	Length in cm	Volume in cc		Total Cl in cc 0.2 N.$AgNO_3$	
		before	after	before	after
intestine	12	0.8	0.65	2.0	2.0
stomach	8	0.4	0.3	1.0	1.2
intestine	10	1.3	1.05	3.3	3.5
intestine	12	4.0	(2.2)	10.1	10.6
intestine	12	1.0	0.8	2.5	2.5

2. Seawater outside, seawater + 0.8 % NaCl inside. Duration of the experiments 6 hours.

Object	Length in cm	Volume in cc		Total Cl in cc 0,2 N.$AgNO_3$	
		before	after	before	after
stomach	16	1.9	2.4	7.3	7.8
stomach	35	4.6	5.7	17.7	18.5
intestine	40	3.9	4.7	15.0	15.1
stomach	35	6.1	7.2	23.4	23.0
stomach	50	15.1	17.8	—	—
intestine	18	1.5	1.9	5.8	5.8

3. Seawater + 0.8 % NaCl outside, seawater inside. Duration of the experiment 7 hours.

Object	Length in cm	Volume in cc		Total Cl in cc 0.2 N.$AgNO_3$	
		before	after	before	after
stomach	30	8.5	7.0	27.4	28.0
intestine	12	1.0	0.8	3.2	3.1
intestine	20	3.9	3.2	12.6	10.5
intestine	14	1.1	0.9	3.5	3.6

4. Diluted seawater outside, seawater inside. Duration of the experiments 5 hours.

Object	Length in cm	Volume in cc		Total Cl in cc 0.2 N.$AgNO_3$	
		before	after	before	after
stomach	12	4.9	5.2	15.8	15.3
intestine	38	4.9	5.7	15.8	16.5
stomach	17	4.0	4.6	12.9	13.2

Alternatively I put them inside or outside in the following combinations :

1. sea-water outside, diluted sea-water inside ;
2. sea-water outside, sea-water + 0.8 % NaCl inside;
3. sea-water + 0.8 % NaCl outside, sea-water inside ;
4. diluted sea-water outside, sea-water inside.

The next tables demonstrate the change in volume and salt-concentration, occurring by the action of the living intestinal wall.

We notice the volume to change as might be expected from the difference in osmotic pressure. The experiments show the water to move in one direction, the salt percentage remaining nearly the same. From these data I conclude that under these experimental conditions, the living wall of the stomach or gut behaves like a membrane impermeable for *chlorides* but not for *water*.

With regard to salt, we have found absolute semi-permeability. The question now arises whether the same impermeability also exists for other substances. I therefore made analogous experiments with the following substances, dissolved in sea-water; corrections having been made for changes in osmotic pressure : glucose, urea, methylen-blue and trypan-blue.

I. *Experiments on glucose.*

In order to avoid pathological conditions and to keep the experiments as "natural" as possible it seemed advisable to use as low concentrations as possible of the "abnormal" substances.

Solutions of glucose in distilled water, isotonic with sea-water, penetrate

the gut wall almost at once (they contain 21—22 %). In the case of a 1 % solution obtained by mixing isotonic glucose in distilled water with sea-water, I hardly was able to demonstrate a reducing action of the surrounding fluid on FEHLING. I also made some experiments using a percentage of 0.4 % in the gut contents and tried to study quantitatively the permeation if there were such. However, it proved to be difficult to estimate with any degree of accuracy the very small quantities passing from inside to outside. I could demonstrate no permeation lying outside the limit of error. In some experiments in which I only quantitatively controlled the permeation of small percentages, the results were always negative too, as long as the organ might be supposed to be living.

If under physiological conditions a permeability for glucose exists, it certainly does not amount much.

II. *Experiments on urea.*

I now filled the guts with solutions of urea in sea-water (0.4 and 0.8 %). The increase in osmotic pressure with regard to seawater is negligeable. At the end of the experiment I determined the concentration of urea in the outside fluid by means of the xanthydrol method of FOSSE (5). The results are shown in the next table.

Object	Length in cm	Volume of filling in cc	Total urea: Before exp. inside in mg.	Total urea: After exp. outside in mg.	Time in min.
stomach	20	2.5	10.0	0	170
stomach	12	2.6	10.4	0	120
stomach	10	2.3	9.0	0	110
stomach	—	4.1	32.8	0	210
intestine	—	2.9	23.2	0	210

The quantities of urea disappearing during 3½ hours from a 0.8 % solution under perfectly physiological conditions cannot be measured, even by this accurate method. In other words, the gut is impermeable for this substance.

III. *Experiments with dyes.*

Dyes have the advantage of a very slight toxicity, furthermore they can be demonstrated in extremely small concentrations : I used trypane-blue and methylen-blue (the former in colloidal solution).

In some of these experiments I did not remove the vessels from the intestinal tract, in order to keep conditions as natural as possible. The percentage of the sea-water solution used amounted to 0.1 %.

Under the conditions of previous experiments I never saw a permeation

of the dye into the surrounding sea water. As long as the pieces of gut might be supposed to be living (judging from their appearance and from the presence of peristalsis) the dye remained in their interior. When, however, the experiment was continued for too long a time e.g. 24 hours, the surrounding fluid soon became blue. I, therefore, conclude that the gut epithelium as long as it is living does not allow these two dyes to pass.

For all substances investigated we have demonstrated the impermeability of the living gut wall. One might ask, whether this is also a property of the dead gut wall or the gut wall in pathological condition.

Experiments on poisoned gut walls are hardly possible without changing the structure of its epithelium. If, in order to kill it, one uses sublimate, alcohol, formaldehyde, or sodium fluoride, it soon becomes very brittle and detached from the rest of the gut wall. Experiments on such organs are practically useless, because now this epithelium forms the limit between lumen and surrounding fluid; the lacunes behind it communicating with the muscular system on various places where the vessels have been torn away.

I therefore preferred to injure the epithelium by other substances, e.g. by high concentrations of glucose. Some parts of the intestinal tract were filled with: 1. a glucose solution of 18 %; 2. equal parts of 18 % glucose and sea-water; 3. a mixture of one part 18 % glucose and three parts sea-water. The results are summarized in the next table:

Object	Time in min.	Volume of filling		Total glucose		Total Cl in cc 0.2 N.$AgNO_3$	
		before cc	after cc	before mg	after mg	before	after
intestine	140	5.0	8.8	900	557	0.0	21.0
stomach	135	5.0	7.8	900	670	0.0	17.0
intestine	125	5.0	—	450	298	8.0	18.5
stomach	105	3.6	4.1	160	81	8.6	12.0

It is evident that these solutions, although nearly isotonic, but very differently composed destroy the vital conditions of the epithelium of stomach and intestines, and thereby abolish the existing impermeability. By the non-physiological differences in concentration between the glucose inside and the NaCl outside, the gut is changed into a dialysing tube. The glucose now permeates very easily as do the chlorides.

From the preceding experiments I conclude that the remarkable properties attributed by COHNHEIM to the gut wall (he supposed it to be capable of active resorption of isotonic solutions and to behave otherwise with regard to solved substances as a diffusion membrane) are to be explained by pathological changes. I believe that he ran his experiments during too long a lapse of time, and that he chose too high concentrations of the solved

substances. The same reproach has been made by ENRIQUES and never an answer has been given, either by COHNHEIM or by other investigators.

Perhaps either the language in which ENRIQUES wrote his paper, or his really severe criticism, perhaps also his parodoxical results, have prevented that his conclusions were accepted. Nevertheless the results of my more extensive experiments plead entirely in favour of his view.

Finally the question arises as to how nutritive substances are utilised in the animal's metabolism, if in fact the living gut wall neither actively resorbs, nor passively allows solved substances to permeate.

The experimental answer to this question is very difficult, owing to the fact, that the animals, 1. if kept in aquaria never take any food, and therefore surely not indicator-solutions; 2. autotomise their intestinal tract as soon as one tries to handle them.

Nevertheless I have been able to get some information with regard to this question by experiments on surviving guts. My experimental procedure was as follows:

From some animals which might be assumed to be in full digestion and resorption, I took the whole intestinal tract, ligated it distally and without any further operation put it into its own body-fluid, previously procured. During the experiment a fine current of O_2 was led through. With a fine pipet I then added a few ccm of a concentrated solution of trypane-blue or methylen-blue in sea-water to the digestive fluid which is always present in the stomach; then I ligated the proximal end. In these experiments I saw the dye mix slowly with the contents by the peristalsis of the stomach. After some hours the contents were collected in order to study changes in intensity of the coloration, if such there were. By means of a binocular microscope I studied the wall and its surroundings, while still showing peristaltical movements.

In two such experiments with trypane-blue I saw no changes; in the two other experiments with methylen-blue I saw a remarkable migration of the dye. In these cases the contents were either completely or almost colourless; the dye had accumulated in some aggregates of amoebocytes, which had become dark blue. But also the stomach wall (not the wall of the intestine sensu stricto) had assumed a bluish shade. Microscopic observation showed that this colour was caused by large quantities of blue amoebocytes, which were found in the lacunes, behind the epithelium itself. The epithelium itself, however, showed hardly any coloration. At some places where the lacunes communicated with the ventral vessel the blue amoebocytes were visible in both. I never saw a diffuse blue coloration. The dye had accumulated everywhere in the phagocytes. Just as in the previous experiments I saw no coloration in the surrounding fluid. Trials to fixate the organs in this condition were failures. The dye cannot be fixated and withdrawal of water from these highly watery tissues causes a contraction and disappearance of the lacunes in the stomach wall.

Very incomplete and full of new problems as my results may be, I

nevertheless believe that they give us definite information with regard to the way in which solved substances may leave the stomach lumen and arrive in the vascular system. They give reason for the believe that other solved substances may also reach the phagocytes behind the epithelium and then be transportated by them. In this way the Holothurian gut would be the realisation of a remarkable possibility in the comparative physiology of digestion and show once more how far removed the Echinoderms are from the other Invertebrates, with regard to structure and functions.

In a paper published elsewhere (6) on the physiology of digestion in Holothurians I dealt more extensively with these and related problems in connection with the utilisation of the food; I hope to be able to collect as soon as possible further data on this remarkable mode of resorption.

The results of the experiments here described may be summarised in the following

CONCLUSIONS.

1°. In the surviving gut of sea-cucumbers no active resorption of isotonic contents takes place.

2°. The living wall of stomach and gut possesses an absolute semi-permeability. Water passes readily, whereas a highly diminished permeation can be demonstrated in the case of other substances, studied in physiological concentrations.

3°. The dead or pathological gut-wall is a simple diffusion membrane.

4°. The omnipresent amoebocytes probably play an important role in the resorption of solved substances.

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Mathematics. — *"Ueber den höheren Zusammenhang von kompakten Räumen und eine Klasse von Abbildungen, welche ihn ungeändert lässt."* By Dr. L. VIETORIS. (Communicated by Prof. L. E. J. BROUWER).

(Communicated at the meeting of May 29, 1926).

Wir wollen im Folgenden den Begriff des mehrfachen, beliebig vieldimensionalen Zusammenhangs der kombinatorischen Topologie auf beliebige kompakte metrische Räume übertragen und zeigen, dass die Zusammenhangsverhältnisse kompakter abgeschlossener Mengen gegenüber eindeutigen stetigen Abbildungen, in welchen jeder Bildpunkt eine Urbildmenge von gewissen einfachsten Zusammenhangsverhältnissen hat, eine gewisse Invarianz besitzen. [1])

I. *Kombinatorische Grundlage.* Wir verstehen unter einem *n-dimensionalen Simplex* $S^{(n)}$ eine Menge von $(n+1)$ Punkten, $\binom{n+1}{2}$ Punktepaaren, $\binom{n+1}{3}$ Punktetripeln, ... und dem einen Punkte-$(n+1)$-tupel, welche man aus ihnen bilden kann. Jedes dieser Punkte-k-tupel heisse eine $(k-1)$-dimensionale Seite von $S^{(n)}$. Für $k = 1,2$ mögen sie Ecken, bezw. Kanten heissen.

Unter einem (*simplizialen*) *Komplex* C verstehen wir eine *endliche* Menge von Simplexen (Teilsimplexen), von denen keines eine Seite des anderen ist, im Verein mit allen Seiten dieser Simplexe. C heisst n-dimensional, wenn n die Dimension seines höchstdimensionalen Teilsimplexes ist; C heisst homogen n-dimensional, wenn alle Teilsimplexe die Dimension n haben.

Dabei lassen wir auch zu, dass ein Komplex ein Teilsimplex mit einer endlichen positiven oder negativen Vielfachheit enthält.

Der Komplex aller $(n-1)$-dimensionalen Seiten eines n-dimensionalen Simplexes heisse der *Rand* desselben. Der Komplex aller jener $(k-1)$-dimensionalen Simplexe eines homogenen k-dimensionalen Komplexes C, welche Seiten einer ungeraden Anzahl von Teilsimplexen von C sind, heisse der *Rand* von C. Ein homogen n-dimensionaler Komplex ohne Rand heisse ein *n-dimensionaler Zyklus.*

Sind $C_1, C_2, \ldots C_k$ homogendimensionale Komplexe derselben Dimension, so verstehen wir unter $C_1 + C_2 + \ldots + C_k$ den Komplex aller in mindestens einem der C_i enthaltenen Teilsimplexe, jedes so oft gezählt

[1]) Diese Untersuchungen gehen von einer mündlichen Bemerkung BROUWERs aus (Vgl. diese Proceedings 29 (1926), S. 445, Anm. 14).

als er in allen C_i zusammengenommen vorkommt. Wir schreiben für zwei homogen n-dimensionale Komplexe K_1, K_2 derselben Dimension $K_1 = K_2$ (mod 2), wenn jedes in K_1 mit einer ungeraden Vielfachheit vorkommende n-dimensionale Simplex auch in K_2 mit ungerader Vielfachheit vorkommt und umgekehrt.

Ist $R^{(k-1)}$ der Rand eines in einem Komplex $C^{(n)}$ enthaltenen Simplexes $S^{(k)}$, so schreiben wir die Homologie $R^{(k-1)} \sim 0$ in $C^{(n)}$. Ferner setzen wir fest:

Gilt für zwei Zykel $T_1^{(k)}$, $T_2^{(k)}$, $T_1^{(k)} \sim 0$ und $T_2^{(k)} \sim 0$ in $C^{(n)}$ und ist $T_1^{(k)} + T_2^{(k)} = T^{(k)}$ (mod 2), dann soll auch $T^{(k)} \sim 0$ in $C^{(n)}$ gelten. Für $T_1^{(k)} + T_2^{(k)} \sim 0$ schreiben wir auch $T_1^{(k)} \sim T_2^{(k)}$.

Sehen wir für $T_1^{(k)} \sim T_2^{(k)}$ in $C^{(n)}$ die Operationen „$+T_1^{(k)}$" und „$+T_2^{(k)}$" als dieselbe Operation an, so erhalten wir eine endliche kommutative Gruppe, welche wir die k-te *Zusammenhangsgruppe* von $C^{(n)}$ nennen.

Die Maximalzahl von in $C^{(n)}$ liegenden k-dimensionalen Zykeln $T_1, T_2, \ldots T_s$, zwischen denen keine Homologie $a_1 T_1 + a_2 T_2 + \ldots + a_s T_s \sim 0$ ($a_i = 0{,}1$) besteht, heisse die k-te *Zusammenhangszahl* von $C^{(n)}$ [2]).

Als 0-te Zusammenhangszahl ergibt diese Definition die um 1 verminderte Komponentenzahl.

Streichen wir in der Definition der obigen Homologien die Festsetzung, dass Zykel, die *modulo* 2 übereinstimmen, homolog sind, so erhalten wir die Homologien POINCARÉ's. Nun bilden die Additionen *orientierter* in $C^{(n)}$ liegender k-dimensionaler Zykel eine Gruppe. Wir nennen sie die k-te *Homologiegruppe* von $C^{(n)}$. Sie ist kommutativ und im allgemeinen unendlich, aber diskret. Die Maximalzahl von in $C^{(n)}$ liegenden *orientierten* k-dimensionalen Zykeln, zwischen denen keine Homologie besteht, heisse die k-te *orientierte Zusammenhangszahl* [3]).

II. *Übertragung auf beliebige kompakte Räume.* Wir sagen, *ein Komplex C liege in einer Menge M*, wenn seine Ecken in M liegen. Ob in M. z. B. zu jeder Kante $[a, b]$ von C, die ja nur als Punktepaar erklärt ist, eine stetige Verbindung zwischen a und b besteht oder nicht, ist uns dabei gleichgültig.

Ist S der Rand eines Simplexes beliebiger Dimension, dessen Kanten $< \varepsilon$ sind, dann schreiben wir $S \sim_\varepsilon 0$ (S ε-homolog 0). Ist $K = S_1 + S_2 + \ldots + S_r$ und $S_i \sim_\varepsilon 0$, so sei auch $K \sim_\varepsilon 0$, ebenso für orientierte, wie für nicht orientierte Zykel. Für $K_1 - K_2 \sim_\varepsilon 0$ schreiben wir auch $K_1 \sim_\varepsilon K_2$.

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[2]) Diese Zusammenhangszahlen sind um 1 kleiner als die „Connectivities" im Sinn von O. VEBLEN, Analysis situs (Cambridge Colloquium 1916), S. 77 ff. Wir haben hier nur die Definition von der Darstellung der Komplexe durch Matrizes losgelöst.

[3]) Sie ist um 1 kleiner als POINCARÉ's k-te BETTIsche Zahl (Journ. Ex. Fol. (2), 1 cah. (1895). Wir folgen in der Zählung dem Vorgang von SCHLÄFLI, KLEIN, DYCK (Siehe die Zitate bei DYCK, Math. Ann. 32 (1888), S. 483) und MANNOURY (Nieuw Archief v. Wiskunde (2) 3 (1898) S. 126-152).

Die unendliche Folge F von k-dimensionalen (nicht orientierten, bezw. orientierten) Zykeln $C_1, C_2, \ldots\ldots$ heisse eine *Fundamentalfolge* in M, wenn die Kantenlänge von C_m mit wachsendem m gegen 0 konvergiert und zu jedem $\varepsilon > 0$ ein n_ε besteht, sodass $C_{n_1} \sim_\varepsilon C_{n_2}$ gilt, sobald $n_1 > n_\varepsilon$ und $n_2 > n_\varepsilon$ sind. F heisse *ε-homolog* 0, wenn es ein n_ε gibt, sodass für $n > n_\varepsilon$ $C_n \sim_\varepsilon 0$ gilt. F heisse eine *Nullfolge* $F \sim 0$, wenn für jedes $\varepsilon > 0$ $F \sim_\varepsilon 0$ gilt.

Sind $\{C_i\}$ und $\{D_j\}$ Fundamentalfolgen (Nullfolgen), so ist auch $\{C_i + D_i\}$ eine Fundamentalfolge (bezw. Nullfolge). $\{C_i\} \sim_\varepsilon \{D_j\}$ und $\{C_i\} \sim \{D_j\}$ bedeute soviel, wie $\{C_i\} - \{D_j\} \sim_\varepsilon 0$, ber. $\{C_i\} - \{D_j\} \sim 0$.

Die Gruppe der Additionen von Fundamentalfolgen heisse die k^{te} *Zusammenhangs-*, bezw. k^{te} *Homologiegruppe* von M.

(1) Die Zusammenhangsgruppen und Homologiegruppen einer beliebigen Menge eines metrischen Raumes können als metrische, und zwar vollständige[4]) Räume aufgefasst werden.

Man hat dazu nur als Abstand zweier Fundamentalfolgen F_1, F_2 die untere Grenze aller Zahlen $\varrho > 0$ zu erklären, für welche $F_1 \sim_\varrho F_2$ ist.

Entsprechend der Endlichkeit der Zusammenhangsgruppe in Komplexen gilt hier:

(2) Die Zusammenhangsgruppe einer kompakten abgeschlossenen Menge M ist kompakt und abgeschlossen.

Denn es gilt zunächst:

(3) In einer kompakten abgeschlossenen Menge M gibt es für festes n und vorgegebenes ε nur endlich viele paarweise nicht ε-homologe nicht orientierte n-dimensionale $\frac{\varepsilon}{3}$-Zykel.

Zum Beweis[5]) wird in M eine endliche Menge A so angenommen, dass jeder Punkt von M von mindestens einem Punkt von A einen Abstand $< \frac{\varepsilon}{3}$ hat. Dann lässt sich zeigen, dass jeder in M liegende $\frac{\varepsilon}{3}$-Zykel C durch endlich viele ε-Abänderungen (Additionen von Simplexrändern $R_i^{(n)} \sim_\varepsilon 0$) in einen in A liegenden Zyklus C^* übergeführt werden kann. Da in A überhaupt nur endlich viele *modulo* 2 verschiedene Zykel liegen, ist damit (3) bewiesen.

Aus (3) folgt fast unmittelbar:

(4) In einer kompakten abgeschlossenen Menge M gibt es zu einem festen n und vorgegebenen $\varepsilon > 0$ nur endlich viele, paarweise nicht ε-homologe Fundamentalfolgen von nicht orientierten, n-dimensionalen Zykeln.

Mit anderen Worten: Die Zusammenhangsgruppe einer kompakten

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[4]) Hausdorff, Grundzüge, S. 315.

[5]) Bezüglich der genauen Ausführung dieses und der folgenden Beweise, verweisen wir auf eine demnächst in den Math. Ann. erscheinende Abhandlung gleichen Titels.

abgeschlossenen Menge ist *total beschränkt*[6]). Da ein total beschränkter vollständiger Raum kompakt ist,[7]) folgt aus (1) und (4) unmittelbar (2).

Für die Homologiegruppe können wir bloss (3′) und (4′) behaupten. Die Beweise sind analog zu denen von (3) und (4).

(3′) Ist M eine kompakte abgeschlossene Menge, so gibt es zu einem festen n und vorgegebenen $\varepsilon > 0$ eine natürliche Zahl s, sodass zwischen irgend s in M liegenden orientierten $\frac{\varepsilon}{3}$-Zykeln $C_1, C_2, \ldots C_s$ eine ε-Homologie $\alpha_1 C_1 + \alpha_2 C_2 + \ldots + \alpha_s C_s \sim_\varepsilon 0$ besteht.

(4′) Ist M eine kompakte abgeschlossene Menge, so gibt es zu einem festen n und vorgegebenen $\varepsilon > 0$ eine natürliche Zahl s, sodass zwischen irgend s Fundamentalfolgen von orientierten in M liegenden n-dimensionalen Zykeln eine ε-Homologie besteht.

Für nicht orientierte Zykel gilt (4′) infolge von (4) erst recht.

Bleibt s, wenn ε gegen 0 abnimmt, beschränkt, so kann man $\limsup s$ als n^{te} (nicht orientierte, bezw. orientierte) Zusammenhangszahl erklären. Die *Ordnung* einer Zusammenhangs-, oder einer Homologiegruppe kann aber endlich, abzählbar unendlich und von der Mächtigkeit des Kontinuums sein.

Wir sagen, die kompakte abgeschlossene Menge M habe eine *p-fache k-dimensionale* (*nicht orientierte, bezw. orientierte*) *Zyklose*, wenn es zu jedem hinreichend kleinen $\varepsilon > 0$ ein $\delta_\varepsilon > 0$ gibt, sodass in M genau p linear ε-unabhängige k-dimensionale (nicht orientierte, bezw. orientierte) δ-Zykel vorkommen, sobald $\delta < \delta_\varepsilon$ ist.[8])

Die Vielfachheit der nicht orientierten Zyklose stimmt, soweit sie endlich ist, mit der Anzahl linear unabhängiger Elemente der Zusammenhangsgruppe überein. Für die orientierte Zyklose lässt sich durch ein Beispiel das Gegenteil zeigen.

III. *Abbildungssätze.* Wir stützen uns im Folgenden auf den Doppelsatz:

(5), (5′) Ist die kompakte abgeschlossene Menge K auf eine andere $\mathfrak{R}$ eindeutig stetig so abgebildet, dass die Menge Y aller Urbilder jedes Punktes y von $\mathfrak{R}$ für $k = 0, 1, 2, \ldots n-1$ eine 0-fache k-dimensionale nicht orientierte, bezw. orientierte Zyklose hat, dann gibt es zu jedem $\varepsilon > 0$ und $\varepsilon' > 0$ ein $\delta > 0$, sodass jeder in $\mathfrak{R}$ liegende n-dimensionale nicht orientierte, bezw. orientierte δ-Zyklus durch ε-Abänderun-

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6) Hausdorff, a. a. O., S. 311.

7) Hausdorff, a. a. O., S. 314 V.

8) Dieser Begriff ist in Analogie zur Vielfachheit der Basis der (eindimensionalen) Zyklosis (BROUWER, Math. Ann. 72, S. 422-425) gebildet. Die Analogie ist insofern nicht vollständig, als sich BROUWERs Basis auf die Fundamentalgruppe, der obige Begriff auf die Zusammenhangs-, bezw. Homologiegruppen bezieht.

gen innerhalb $\mathfrak{R}$ in einen n-dimensionalen ε-Zyklus übergeführt werden kann, der das Bild (mindestens) eines in K liegenden nicht orientierten, bezw. orientierten n-dimensionalen ε'-Zyklus ist.

Der Beweis verläuft für nicht orientierte und orientierte Zykel fast gleich und stützt sich auf den Hilfssatz:

Ist die kompakte abgeschlossene Menge K eindeutig stetig auf $\mathfrak{R}$ abgebildet, so gibt es zu jedem $\varepsilon > 0$ und $\varepsilon' > 0$ ein $\delta > 0$, derart, dass zu jeder Menge $\mathfrak{R}_1 \subseteq \mathfrak{R}$ von einem Durchmesser $< \delta$ ein Punkt y_0 besteht, sodass jeder Punkt x von K, dessen Bild in $\mathfrak{R}_1$ liegt, von der Urbildmenge Y_0 von y_0 einen Abstand $< \varepsilon$ hat und ausserdem y_0 von jedem Punkt von $\mathfrak{R}_1$ weniger als ε' entfernt ist.

Zum Beweis von (5) wird, wenn $\mathfrak{Z}$ der gegebene δ-Zyklus ist, zunächst zu jeder Ecke y_i von $\mathfrak{Z}$ ein Punkt x_i in Y_i beliebig gewählt, dann zu jeder Kante $[y_i, y_j]$ von $\mathfrak{Z}$ zufolge (6) ein Punkt y_{ij} von $\mathfrak{R}$ der von y_i und y_j „mit δ kleine" Abstände hat und für welchen auch jeder Punkt von $Y_i + Y_j$ von Y_{ij} einen mit δ kleinen Abstand hat. Weil Y_{ij} eine 0-fache 0-dimensionale Zyklose hat, d.h. zusammenhängend ist, kann also x_i mit x_j durch eine sonst ganz in Y_{ij} liegende „mit δ feine" Kette K_{ij} von endlich vielen Punkten verbunden werden. Ist $[y_i\ y_j\ y_k]$ eine zweidimensionale Seite von $\mathfrak{Z}$, so ist der Durchmesser des Sechsecks $[y_i\ y_{ij}\ y_j\ y_{jk}\ y_k\ y_{ki}]$ mit δ klein. Nach (6) kann ich in $\mathfrak{R}$ einen Punkt y_{ijk} wählen, der von allen Punkten des Sechsecks mit δ kleine Abstände hat, derart, dass jeder Punkt der Menge $Y_i + Y_{ij} + Y_j + Y_{jk} + Y_k + Y_{ki}$ von Y_{ijk} einen mit δ kleinen Abstand hat. Ich kann also zu jedem Punkt x des eindimensionalen Zyklus $Z_{ijk} = K_{ij} + K_{jk} + K_{ki}$ in Y_{ijk} einen Punkt x' wählen, dessen Abstand von x mit δ klein ist, d.h. zu Z_{ijk} einen eindimensionalen Zyklus Z'_{ijk}. Weil nun Y_{ijk} eine 0-fache eindimensionale Zyklosis hat, ist Z'_{ijk} Rand eines in Y_{ijk} liegenden homogen 2-dimensionalen Komplexes, der mit δ fein ist. Zusammen mit dem „zylindrischen Streifen" zwischen Z_{ijk} und Z'_{ijk}, den wir uns simplizial zerlegt denken, gibt er einen mit δ feinen Komplex K_{ijk}, dessen Rand Z_{ijk} ist. So fahren wir fort, bis wir schliesslich mithilfe der 0-fachen $(n-1)$-dimensionalen Zyklose der Y alle jene $(n-1)$-dimensionalen Zykel, ausgefüllt haben, welche beim Schritt vorher konstruiert worden sind.

Aus (5) lässt sich unter Verwendung der Kompaktheit der Zusammenhangsgruppen ableiten:

(7) Unter den Voraussetzungen von (5) ist die n^{te} Zusammenhangsgruppe $\mathfrak{Z}^{(n)}$ von $\mathfrak{R}$ das Bild der n^{ten} Zusammenhangsgruppe $Z^{(n)}$ von K (bezüglich der zugrunde gelegten Abbildung), und die n^{te} Zusammenhangszahl von $\mathfrak{R}$ ist nicht grösser als die von K[9]).

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[9]) Der zweite Teil von (7) ist eine Verallgemeinerung eines von BROUWER herrührenden, in meiner in Anm. 1 zitierten Abhandlung wiedergegebenen Satzes.

Ob (7) auch für orientierte Zykel gilt, bleibe offen. Dagegen können wir auch für diese behaupten:

(7*b*) Unter den Voraussetzungen von (5′) ist die k^{te} Homologiegruppe von $\mathfrak{R}$ für $k < n$ das Bild der k^{ten} Homologiegruppe von K.

Der Beweis stützt sich auf folgende Sätze (5*b*) und (8), die ähnlich wie (5) und (5′) bewiesen werden.

(5*b*) Unter den Voraussetzungen von (5) bezw. (5′) gibt es für $k < n$ zu jedem $\varepsilon > 0$ und $\varepsilon' > 0$ ein $\delta > 0$, sodass jeder in $\mathfrak{R}$ liegende mit 0 δ-homologe k-dimensionale Zyklus mit einem k-dimensionalen Zyklus in $\mathfrak{R}$ ε-homolog ist, der das Bild eines in K mit 0 ε'-homologen k-dimensionalen Zyklus ist.

(8) Unter den Voraussetzungen von (5), bezw. (5′) ist jede Folge F von in K liegenden k-dimensionalen (nicht orientierten, bezw. orientierten) Zykeln mit gegen 0 konvergierenden Kantenlängen und $k < n$, deren Bild in $\mathfrak{R}$ eine Nullfolge ist, selbst eine Nullfolge.

Auf Grund von (5*b*) und (8) kann man nämlich zu jeder in $\mathfrak{R}$ liegenden Fundamentalfolge von k-dimensionalen Zykeln eine mit ihr homologe Fundamentalfolge finden, welche Bild einer in K liegenden Fundamentalfolge ist. Da nun ausserdem wegen (8) zwei Fundamentalfolgen $F_1 F_2$ von k-dimensionalen Zykeln in K, deren Bilder f_1, f_2 mit einander homolog in $\mathfrak{R}$ sind, homolog in K sind, gilt schliesslich:

(7*c*) Unter den Voraussetzungen von (5), bezw. (5′) ist für $k < n$ die k^{te} Zusammenhangsgruppe, bezw. die k^{te} Homologiegruppe von $\mathfrak{R}$ mit der von K identisch[10]).

Für die Vielfachheit der Zyklosis lässt sich mit ähnlichen Mitteln und zwar in gleicher Weise für orientierte, wie für nicht orientierte Zykel zeigen:

(7*d*) Unter den Voraussetzungen von (5), bezw. (5′), ist für $k < n$ die Vielfachheit der k-dimensionalen nicht orientierten, bezw. orientierten Zyklosis von $\mathfrak{R}$ genau gleich der von K.

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[10]) Bezüglich der analogen Behandlung der *Fundamentalgruppe* verweisen wir auf die in Anm. 5 angekündigte Abhandlung.

Mathematics. — *"Ueber stetige Bilder von Punktmengen."* By Dr. W. HUREWICZ. (Communicated by Prof. L. E. J. BROUWER).

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Im Folgenden untersuchen wir die stetigen Bilder von Punktmengen vom Standpunkt der allgemeinen Dimensionstheorie. [1]) Es gilt diesbezüglich vor allem eine Charakterisierung der n-dimensionalen kompakten Räume: *Jeder kompakte n-dimensionale metrische Raum K lässt sich darstellen als eindeutiges stetiges Bild einer abgeschlossenen nulldimensionalen Menge (insbesondre also einer abgeschlossenen nirgends dichten linearen Menge* [2]*), derart dass jeder Punkt von K Bild von höchstens $n+1$ Punkten ist, — und umgekehrt ist ein eindeutiges stetiges Bild einer kompakten nulldimensionalen Menge, das höchstens $(n+1)$-fache Punkte enthält, höchstens n-dimensional* [3]).

Dieser Satz ist ein Spezialfall einer analogen Charakterisierung für allgemeine separable Räume, inbesondre also für Teilmengen kompakter Räume. Zwar können schon eineindeutige stetige Bilder nulldimensionaler Mengen eindimensional sein [4]), ja beliebig hohe Dimension besitzen. Die Uebertragung der angeführten Charakterisierung auf nicht kompakte Räume gelingt indes durch Einführung folgender Begriffe: Wir nennen zunächst *die mehrdeutige Abbildung A* der Menge M auf die Menge M^* *stetig*, falls zu jedem Punkt p von M und zu jeder Umgebung V der Bildmenge $A(p)$ eine Umgebung $U(p)$ von p in M existiert, so dass alle Bilder von Punkten aus $U(p)$ in V liegen, oder, was gleichbedeutend ist, falls jede in M^* abgeschlossenen Menge ein in M abgeschlossene Urbildmenge [5]) hat.

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[1]) Vgl. MENGER. Bericht über die Dimensionstheorie, Jahresber. d. deutschen Mathem. Vereinig. 35, 1926 und URYSOHN. Mémoire sur les multiplicités Cantoriennes, Fund. Math. VII u. VIII.

[2]) Mit Rücksicht auf den SIERPIŃSKIschen Satz von der Homöomorphie der nulldimensionalen mit linearen Mengen, Fund. Math. II, 1921, S. 89.

[3]) LEBESGUE hat nachgewiesen (Fund. Math. II, 1921, S. 283), dass jede Begrenzung einer offenen Menge des R_{n+1} (variété frontière à n dimensions) sich darstellen lässt als eindeutiges stetiges Bild einer linearen perfekten Menge mit höchstens $(n+1)$-fachen Punkten; da die Begrenzungen offener Mengen des R_{n+1} höchstens n-dimensional sind, ist dies ein Spezialfall des obigen Satzes. Jeder *perfekte* n-dimensionale Raum kann durch eindeutige stetige Abbildung mit höchstens $(n+1)$-fachen Punkten aus der nirgends dichten perfekten CANTOR'schen Menge gewonnen werden. Dass überhaupt jeder kompakte metrische Raum stetiges Bild der CANTOR'schen Menge ist, wurde von ALEXANDROFF (diese Proceedings 28, 1925, S. 997) ausgesprochen; vgl. übrigens eine Bemerkung von MAZURKIEWICZ (Fund. Math. I, 1920, S. 179—180).

[4]) Vgl. das einfache Beispiel von MENGER, Ueber die Dimension von Punktmengen II, Monatshefte f. Math. u. Phys. 34, 1924, S. 141.

[5]) Unter der Urbildmenge eines Teiles N von M^* verstehen wir die Gesamtheit aller Punkte von M, die mindestens einen in N liegenden Bildpunkt besitzen.

Wir nennen *eine eindeutige Abbildung beiderseits stetig*, wenn sie stetig ist und ihre (im Allgemeinen mehrdeutige) Umkehrung im definierten Sinn ebenfalls stetig ist.[1]) Es gilt nun das folgende

Theorem. *Unter den separabeln metrischen Räumen sind die n-dimensionalen dadurch charakterisiert, dass sie sich erstens als eindeutige beiderseits stetige Bilder einer nulldimensionalen Menge (also auch einer linearen Menge ohne Teilintervall) mit höchstens $(n+1)$-fachen Punkten darstellen lassen, während sie sich zweitens nicht als eindeutige beiderseits stetige Bilder einer nulldimensionalen Menge darstellen lassen, so dass jeder Punkt weniger als $n+1$ Urbildern zugeordnet ist.*

Den Beweis der ersten Hälfte des Theorems stützen wir darauf, dass in jedem n-dimensionalen separabeln Raum R ein System von abgeschlossenen Mengen[2]) mit folgenden Eigenschaften existiert:

1. Es gibt endlich viele, etwa ν_1, in R abgeschlossene Mengen $R_1, R_2, \ldots R_{\nu_1}$ "des ersten Schrittes", und $R = \sum_{k=1}^{\nu_1} R_k$.

2. Wenn $R_{i_1, i_2, \ldots i_n}$ eine bereits definierte Menge des Systems ist, dann gibt es endlich viele, etwa $\nu_{i_1, i_2, \ldots i_n}$ abgeschlossene Mengen des $(n+1)^{\text{-ten}}$ Schrittes, $R_{i_1, i_2, \ldots i_n, k}$, und $R_{i_1, i_2, \ldots i_n} = \sum_{k=1}^{\nu_{i_1, i_2, \ldots i_n}} R_{i_1, i_2, \ldots i_n, k}$.

3. Jede Folge von in einander geschachtelten Mengen $R_{i_1}, R_{i_1, i_2}, \ldots$ $\ldots R_{i_1, i_2, \ldots i_n}, \ldots$ zieht sich entweder auf einen Punkt zusammen oder hat einen leeren Durchschnitt.

4. Für jedes k sind je $n+2$ Mengen des $k^{\text{-ten}}$ Schrittes fremd.

(Dieser Satz ist eine Verallgemeinerung des von MENGER[3]) und URYSOHN[4]) bewiesenen Satzes von der Zerlegbarkeit kompakter n-dimensionaler Räume in endlich viele beliebig kleine abgeschlossene Teile, die zu je $n+2$ fremd sind. Wie MENGER bemerkt hat[5]), gilt dieser letztere Satz auf Grund eines Ergebnisses von mir[6]) für beliebige Teilmengen kompakter Räume.)

Wir ordnen nun jeder Menge $R_{i_1, i_2, \ldots i_k}$ des $k^{\text{-ten}}$ Schrittes ein abgeschlossenes Interval $I_{i_1, i_2, \ldots i_k}$ von einer Länge $< \frac{1}{k}$ im Innern des Intervalls $[0,1]$ zu, so dass *erstens* je zwei Intervalle mit gleich vielen Indizes

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[1]) Jede eindeutige stetige Abbildung eines kompakten Raumes ist beiderseits stetig.

[2]) Vgl. den verwandten Begriff des finiten Umgebungssystems bei MENGER, diese Proceedings **29**.

[3]) Monatshefte f. Math. u. Phys. 34, 1924, S. 153.

[4]) Fund. Math, VIII.

[5]) Vgl. MENGER, Bericht.

[6]) Nämlich, auf Grund des Satzes, dass n-dimensionale Teilmengen kompakter Räume in endlich viele beliebig kleine relativ abgeschlossene Teile zerlegbar sind, die zu je zwei höchstens $(n-1)$-dimensionale Durchschnitte haben. Vgl. meine Arbeit „Normalbereiche und Dimensionstheorie", Mathem. Annalen 96.

fremd sind und *zweitens* stets $I_{i_1, i_2, \dots i_{k-1}, i_k}$ im offenen Intervall $I_{i_1, i_2, \dots i_{k-1}}$ liegt. Sodann betrachten wir auf der Strecke die (nulldimensionale) Menge aller Punkte p, auf die sich eine Folge von Intervallen $I_{i_1}, I_{i_1, i_2}, \dots I_{i_1, i_2, \dots i_n}, \dots$ zusammenzieht, so dass die Folge der entsprechenden Mengen R_{i_1}, $R_{i_1 i_2}, \dots R_{i_1, i_2, \dots i_n} \dots$ einen nicht leeren Durchschnitt hat und sich mithin auf einen Punkt p^* von R zusammenzieht. Wir ordnen nunmehr p^* dem Punkt p als Bildpunkt zu. Dadurch ist, wie man leicht einsieht, eine eindeutige beiderseits stetige Abbildung einer linearen nulldimensionalen Menge auf den vorgegebenen Raum R definiert, wobei jeder Punkt von R höchstens $n+1$ Urbilder besitzt.

Den Beweis der zweiten Hälfte unseres Theorems stützen wir auf den folgenden vielleicht auch an sich interessanten Satz:

Ist der Raum R^* eindeutiges beiderseits stetiges Bild des separabeln n-dimensionalen metrischen Raumes R, so zwar dass alle Punkte von R eine gleiche endliche Anzahl von Urbildern besitzen, dann ist auch R n-dimensional.

An dieser Stelle zeigen wir, da es für das Folgende ausreicht, bloss, dass R^* höchstens n-dimensional ist[1]. Zum Beweise betrachten wir ein abzählbares System S von Umgebungen in R, so dass sich auf jeden Punkt von R eine Umgebungsfolge aus S zusammenzieht. Bezeichnet k die für alle Punkte von R^* gemeinsame Anzahl von Urbildern in R, so ordnen wir alle Systeme von je k Umgebungen aus S, die samt ihren Begrenzungen paarweise fremd sind, in eine abzählbare Folge $K_1, K_2, \dots K_m, \dots$ Mit R^*_m bezeichnen wir die in R^* abgeschlossene Menge, welche Durchschnitt ist von den Bildern der k Umgebungen aus S, die in K_m zusammengefasst sind. Man überzeugt sich sofort davon, dass $R^* = \sum\limits_{m=1}^{\infty} R_m$ gilt. Ferner ist jede Menge R^*_m höchstens n-dimensional. Denn betrachten wir irgend eine bestimmte von den k Umgebungen, aus denen K_m zusammengesetzt ist, und ordnen wir jedem Punkt von R^*_m dasjenige seiner Urbilder zu, welches in dieser Umgebung liegt, — so haben wir, wie man leicht zeigt, eine umkehrbar eindeutige beiderseits stetige (also topologische) Abbildung zwischen R^*_m und einem Teil der n-dimensionalen Menge R. — Demnach ist R^* Summe von abzählbar vielen in R^* abgeschlossenen höchstens n-dimensionalen Mengen, also[2] auch selbst höchstens n-dimensional.

Eine unmittelbare Folgerung hieraus lautet:

Ist R^* eindeutiges beiderseits stetiges Bild des separabeln nulldimensionalen Raumes R, dann ist für jede natürliche Zahl k die Menge aller Punkte

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[1]) Eine Erniedrigung der Dimension kann, nebenbei bemerkt, durch eine eindeutige beiderseits stetige Abbildung schon dann nicht hervorgerufen werden, wenn jeder Punkt der Bildmenge eine nulldimensionale Menge von Urbildern besitzt.

[2]) Vgl. meine oben zitierte Arbeit und TUMARKIN, diese Proceedings 28, S. 994.

von R^*, die genau k Urbilder in R besitzen, entweder leer oder nulldimensional [1]).

Da die Summe von $n+1$ nulldimensionalen Mengen höchstens n-dimensional ist [2]), folgt aus dem Bewiesenen:

Wenn der n-dimensionale Raum R^* eindeutiges beiderseits stetiges Bild des separabeln nulldimensionalen Raumes R ist, so dass jeder Punkt von R^* von endlicher Vielfachheit ist, d.h. nur endlich viele Urbilder in R besitzt, dann kommen in R^* Punkte von mindestens $n+1$ verschiedenen Vielfachheiten vor.

In dieser Behauptung ist die zweite Hälfte unseres Theorems enthalten.

Die bisherigen Resultate lassen sich teilweise auf die eindeutigen stetigen Bilder der n-dimensionalen Mengen, insbesondre auf die stetigen Bilder der Strecke (die Jordanschen Kurven) übertragen. Zwar gilt, wie man sich durch einfache Beispiele klar macht, im allgemeinen nicht, dass sich ein n-dimensionales im kleinen zusammenhängendes Kontinuum als eindeutiges stetiges Bild der Strecke mit höchstens $(n+1)$-fachen Punkten darstellen lässt. [3]) Es gilt aber, *dass bei jeder eindeutigen stetigen Abbildung der Strecke auf einen Raum R^* die Menge aller Punkte von einer gegebenen endlichen Vielfachheit nulldimensional ist, wofern es in R^* keinen einfachen Bogen gibt, der einen offenen Teil von R^* vollständig ausfüllt.* Ist also der Bildraum n-dimensional und jeder seiner Punkte von endlicher Vielfachheit, so müssen Punkte von mindestens $n+1$ verschiedenen Vielfachheiten auftreten, insbesondre also auch Punkte von mindestens $(n+1)$-facher Vielfachheit. [4]) Weiter ergibt sich, *dass bei jeder eindeutigen stetigen Abbildung der Strecke* [5]) *auf ein Intervall des R_n die Menge aller Punkte von einer Vielfachheit $\geqq k$ $(k = 1, 2, \ldots n)$ zusammenhängend ist* [6]).

Eine ausführliche Darstellung der Resultate dieser Arbeit erscheint demnächst in den Mathematischen Annalen.

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[1]) Es gilt allgemein, dass in einem eindeutigen beiderseits stetigen Bild eines separabeln n-dimensionalen Raumes für jedes k die Menge aller k-fachen Punkte höchstens n-dimensional ist. Die Menge aller höchstens k-fachen Punkte ist stets ein G_δ.

[2]) Vgl. URYSOHN, Fund. Math. VIII.

[3]) Für Intervalle des R_n gilt dies wohl, wie LEBESGUE, Fund. Math. II. 1921, S. 280 gezeigt hat.

[4]) Die Existenz mindestens $(n+1)$-facher Punkte bei der Abbildung der Strecke auf ein Intervall des R_n wurde von LEBESGUE a.O. S. 279 bewiesen.

[5]) Der Satz gilt auch für die Abbildungen von nulldimensionalen Mengen.

[6]) Von dieser Menge kann man ferner zeigen, dass sie (und zwar in jedem ihrer relativ offenen Teile) ein $(n+1-k)$-dimensionales Teilkontinuum enthält. Dass jede Jordankurve, die ein Quadrat ausfüllt, ein Kontinuum von Doppelpunkten enthält, wurde bereits von HAHN bewiesen. Vgl. Annali di Mathem., 1913, S. 48.